PACKING CLOSED TRAILS INTO DENSE GRAPHS

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ABSTRACT. It has been shown [Balister, 2001] that if n is odd and $m_1, \ldots, m_t$ are integers with $m_i \geq 3$ and $\sum_{i=1}^{t} m_i = |E(K_n)|$ then K_n can be decomposed as an edge-disjoint union of closed trails of lengths $m_1, \ldots, m_t$. Here we show that the corresponding result is also true for any sufficiently large and sufficiently dense even graph G.

1. INTRODUCTION

All graphs considered will be finite simple graphs. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. As usual $\delta(G)$ will denote the minimum degree of G. We say G is *even* if the degree $d_G(v)$ of every vertex $v \in V(G)$ is even. We shall usually write $n = |V(G)|$ for the number of vertices of G. If $S \subseteq E(G)$, then we write $G \setminus S$ for the graph with the same vertex set as G, but edge set $E(G) \setminus S$. Sometimes we shall abuse notation by writing, for example, $G \setminus H$ for $G \setminus E(H)$ when H is a subgraph of G.

The main result we shall prove is the following.

Theorem 1. *There exist absolute constants N and $\epsilon > 0$ such that for any even graph G on n vertices with $n \geq N$ and $\delta(G) \geq (1-\epsilon)n$ and for any collection of integers $m_1, \ldots, m_t$ with $m_i \geq 3$ and $\sum_{i=1}^{t} m_i = |E(G)|$ one can write G as the edge-disjoint union of closed trails $C_1, \ldots, C_t$ with C_i of length m_i. In addition, given any fixed $v \in V(G)$, we can also ensure that C_1 meets v.*

It is worth noting that the ϵ given by the proof of Theorem 1 is extremely small due to the use of a result of Gustavsson [6] which also needs a very small ϵ.

In [2] this theorem was proved when $G = K_n$ and n odd, and when $G = K_n - I$ and n even, where I is a 1-factor of K_n. In contrast to Theorem 1, this holds even for small n. These results are closely related to Alspach's Conjecture [1] which asks whether $G = K_n$ or $K_n - I$ can be decomposed into *cycles* of lengths $m_1, \ldots, m_t$. Some results on this problem are given in [3].

The strategy of the proof of Theorem 1 will be to first pack closed trails of arbitrary lengths into graphs formed by linking octahedra ($K_{2,2,2}$) together. This is done in Section 2. These linked octahedra can be formed by taking a trail of linked triangles and doubling up the vertices (see Figure 1). Section 3 shows that the triangles in any triangle decomposition of a dense graph can be ordered in such a way to form such a trail of linked triangles. In Section 4 we show how to use these results to prove Theorem 1 when n is even by reducing to the case when the graph is obtained by doubling the vertices in a graph formed by such a trail of triangles. The proof is extended to the case when n is odd in Section 5.

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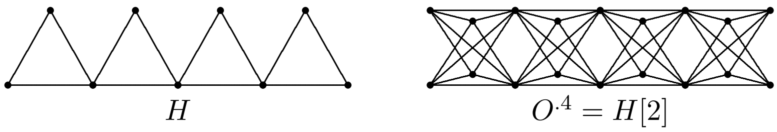


FIGURE 1. Trails of triangles and octahedra.

2. Packing Octahedra

If G_1 and G_2 are graphs, define a *packing* of G_1 into G_2 as a map $f\colon V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map on edges $xy \mapsto f(x)f(y)$ is a bijection between $E(G_1)$ and $E(G_2)$. Note that f is *not* required to be injective on vertices. Hence if G_1 contains a path or cycle, its image in G_2 will be a trail or closed trail. With this notation, the problem is one of packing a disjoint union of cycles $\bigcup_{i=1}^{t} C_{m_i}$ into some dense even graph G.

We shall define for some graphs, *initial* and *final* links as (ordered) pairs of vertices, (possibly the same pair). If these have been defined for G_1 and G_2, then we write $G_1 \cdot G_2$ for the graph obtained by identifying the final link of G_1 with the initial link of G_2 (in the same order). The graph $G_1 \cdot G_2$ will be undefined if an edge occurs in both these links. The initial link of the resultant graph will be that of G_1 and the final link will be that of G_2. This makes $\cdot$ into an associative operation on such graphs when defined. Similarly, the initial link of $G_1 \cup G_2$ will be that of G_1 and the final link will be that of G_2. We shall also write $G^{\cdot n}$ for $G \cdot G \cdots G$ and $G^{\cup n}$ for $G \cup \cdots \cup G$ when there are n copies of G. If H is a graph, denote by $H[2]$ the graph obtained by replacing each vertex $v \in V(H)$ by a pair of vertices v_1, v_2, and each edge $uv \in E(H)$ by four edges u_iv_j, $1 \le i, j \le 2$.

As in [2], let $O = K_{2,2,2} = C_3[2]$ be the graph of an octahedron. This graph is tripartite with three vertex classes, each class consisting of two vertices. The first vertex class will be the initial link of O and the last vertex class will be the final link of O. In fact by symmetry it does not matter which vertex classes are chosen, or the order of the vertices in either link.

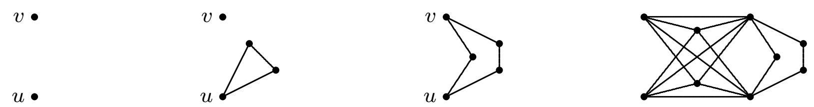
Hence for $n \ge 1$, $O^{\cdot n}$ represents a graph on $4n + 2$ vertices obtained by taking n octahedra and identifying a pair of vertices of the ith octahedron with a pair in the $(i+1)$th octahedron. Note that $O^{\cdot n} = H[2]$ where H is the graph obtained by joining n triangles together along a path (see Figure 1).

For a path P_n of edge length n with endpoints u and v, make (u, v) both the initial and final link of P_n. The graph $P_{a_1,\dots,a_r} = P_{a_1} \cdot P_{a_2} \cdots P_{a_r}$ will be a graph with specified link vertices (u, v) consisting of independent paths of length a_i from u to v. In the special case when $r = 0$ we write $P_\emptyset$ for the empty graph on $\{u, v\}$. Write $C'_n = C_n \cup E_1$ for a cycle of length n together with an extra independent vertex. The pair (u, v) will be the initial and final link of C'_n where v is the independent vertex and u is any vertex of the cycle.

Definition 1. *Define the graphs L_n for $n = 0$ and $n \ge 3$ by*

$$L_0 = P_\emptyset, \quad L_3 = C'_3, \quad L_4 = P_{2,2}, \quad L_5 = P_{2,3}, \quad \textit{and} \quad L_n = P_{4,n-4} \textit{ for } n \ge 6,$$

except that L_6 will be defined as either $P_{4,2}$ or $P_{3,3}$ *and L_8 will be defined as* either $P_{4,4}$ or $P_{2,2,2,2}$.



$L_0 = P_\emptyset$ $\quad$ $L_3 = C_3'$ $\quad$ $L_5 = P_{2,3}$ $\quad$ $O \cdot L_5$

FIGURE 2. Examples of Graphs L_n and $O \cdot L_n$.

Note that we can pack C_n into L_n for all $n > 0$. Note also that the definition of L_8 differs slightly from that in [2].

We now need some simple packing results from [2] which we summarize here. Special care must be taken with the graphs L_6 and L_8. Whenever we say there is a packing of L_m into some other graph, then this is true with *either* choice of L_m when $m = 6$ or 8. On the other hand, if we say there is a packing into some graph involving L_m, then we mean only that a packing exists for *some* choice of L_m. We quote the following result.

Theorem 2 (Theorem 14 of [2])**.** *Suppose that either $m + \sum m_i \geq 15$ or $m + \sum m_i = 12$ with $m \geq 0$, $m \neq 1, 2$, $m_i \geq 5$, $m_i \neq 6$. Then for some subset S and some m' we can pack $L_m \cup (\bigcup_{i \in S} C_{m_i})$ into $O \cdot L_{m'}$ with the initial link of L_m mapped to the initial link of $O \cdot L_{m'}$, except in the cases when $m \in \{0, 4, 5, 9\}$ and all the m_i are equal to 5.*

We shall also need:

Lemma 3. *The following packings exist.*

$L_3 \cup C_3 \cup C_3 \cup C_3$	*into*	O
$L_3 \cup C_3 \cup C_6$	*into*	O
$L_4 \cup C_3 \cup C_5$	*into*	O
$L_6 \cup C_3 \cup C_3$	*into*	O
$L_6 \cup C_6$	*into*	O
$L_9 \cup C_3$	*into*	O
$C_4 \cup C_4 \cup C_4$	*into*	O
$C_5 \cup C_5 \cup C_5 \cup C_5$	*into*	$O \cdot L_8$
$L_4 \cup C_5 \cup C_5 \cup C_5 \cup C_5$	*into*	$O \cdot O$

In all relevant cases the initial link of L_m is mapped to the initial link of the resulting graph.

Proof. Each of these packings can be constructed easily by hand, however we shall construct them using the results of [2]. The graph O can be packed with four triangles, at least one of which meets the first vertex of the initial link and hence forms an L_3. Therefore the first packing listed above exists. By Lemma 13 of [2] we have packings of $L_n \cup L_{12-n}$ $(3 \leq n \leq 9)$ and $L_n \cdot C_3' \cup L_{9-n}$ $(4 \leq n \leq 6)$ into O. By symmetry we also have packings of $L_{9-n} \cup C_3' \cdot L_n$ $(4 \leq n \leq 6)$ into O. These packings give the next five packings above (using the fact that C_n can be packed into L_n or C_n' and $G_1 \cup G_2$ can be packed into $G_1 \cdot G_2$). We can pack two C_4s into $P_{2,2,2,2}$ by pairing up the paths. Hence we have a packing of three C_4s into $P_{2,2,2,2} \cup P_{2,2}$. Once again, this can be packed into O by Lemma 13 of [2]. This lemma also gives a packing of $L_0 \cup P_{3,3,3,3}$ into O. We can pack four C_5s into $P_{3,3,3,3} \cdot P_{2,2,2,2}$ by joining paths of length 2 with paths of length 3. Hence we have a packing of four C_5s into $O \cdot P_{2,2,2,2}$. Finally L_4 and four C_5s can be packed into

$L_4 \cup P_{2,2,2,2} \cdot O$ which can then be packed into $O \cdot O$ using the $P_{2,2} \cup P_{2,2,2,2}$ packing mentioned above. □

Theorem 4. *If* $\sum_{i=1}^{t} m_i = 12n$ *and* $m_i \geq 3$ *then one can decompose* $O^{\cdot n}$ *as an edge-disjoint union of closed trails of lengths* $m_1, \dots, m_t$.

Proof. We need to pack $\bigcup_{i=1}^{t} C_{m_i}$ into $O^{\cdot n}$. If we have three C_4s then we can pack these into the first O using Lemma 3. The result will then follow by induction on n. Similarly if we have four C_3s or two C_6s or $C_3 \cup C_3 \cup C_6$ then we can pack these into the first O and use induction. Hence we may assume there are at most two C_4s and the C_3s and C_6s have total length at most 9.

If we have two C_4s, pack $C_4 \cup C_4$ as $L_8 = P_{2,2,2,2}$. If we have one C_4, pack it as L_4. Otherwise start with L_0. If we temporarily discard C_3s and C_6s, the total length of cycles will still be at least $12(n-1)+3$.

Now we pack the other cycles, which are all of length 5 or ≥ 7. We can assume that we have already packed an $O^{\cdot b} \cdot L_m$ with $0 \leq b < n-1$. If there are some C_5s remaining, we can also assume $m \in \{0, 4, 6, 8\}$. We shall pack the remaining cycles inductively into graphs of the same form with larger values of b, starting with the C_5s. If we have enough remaining C_5s use the packings

$$\begin{array}{lll} L_0 \cup C_5 \cup C_5 \cup C_5 \cup C_5 & \text{into} & O \cdot L_8 \\ L_4 \cup C_5 \cup C_5 \cup C_5 \cup C_5 & \text{into} & O \cdot O \\ L_6 \cup C_5 \cup C_5 & \text{into} & O \cdot L_4 \\ L_8 \cup C_5 \cup C_5 & \text{into} & O \cdot L_6 \end{array}$$

The first two are from Lemma 3, the last two from Theorem 2. In each case the initial links match, so we can pack $O^{\cdot b} \cdot L_m$ into $O^{\cdot b+1} \cdot L_{m'}$, $m' \in \{4, 6, 8\}$, or $O^{\cdot b+2} = O^{\cdot b+2} \cdot L_0$.

Assume we have enough C_5s to reach a total length of at least $12(n-1)+3$. We shall pack at least $n-1$ of the Os completely, except when we have packings of $O^{\cdot n-2} \cdot L_m$ with $m = 0$ or 4 and at least three more C_5s. If $m = 4$ we must have four remaining C_5s ($24 - 4 - 5 - 5 - 5 = 5$ edges are left for the remaining cycles), so can use the L_4 packing above to finish. If $m = 0$ the remaining cycle(s) are of total length 9. We deal with each case separately. Recall that we can always pack C_{m_i} into L_{m_i}. If there is a C_3, use the packing of $L_3 \cup C_5 \cup C_5 \cup C_5$ from Theorem 2. If there is another C_5, pack the four C_5s into $O \cdot L_8$ using Lemma 3. If there is no C_3 or C_5 then there is just one remaining C_9, in which case use the $L_5 \cup C_9 \cup C_5$ packing from Theorem 2. Hence in all cases we have packed some graph of the form $O^{\cdot n-1} \cdot L_m$ (or $O^{\cdot n}$).

Now assume we do not have enough C_5s to pack $n-1$ octahedra. Hence there is at least one cycle of length at least 7. After packing as many C_5s as possible, we shall have at most one C_5 left or three C_5s if we have packed $O^{\cdot b} \cdot L_0$ or $O^{\cdot b} \cdot L_4$. Pack $L_0 \cup C_5$ into L_5 or $L_4 \cup C_5 \cup C_m$ into $O \cdot L_{m-3}$ (for some $m \geq 7$) to ensure we have at most two C_5s left. Now continue packing the remaining cycles using Theorem 2. Whenever we have packed $O^{\cdot b} \cdot L_m$ with $b < n-1$, we have enough extra cycles to pack L_m and some cycles into $O \cdot L_{m'}$ with initial link matching by Theorem 2, and hence we can pack $O^{\cdot b+1} \cdot L_{m'}$. The only exception is when we try to pack $L_9 \cup C_5 \cup C_5$ into the $(n-1)^{\text{st}}$ O. (There are at most two C_5s and the other combinations not allowed by Theorem 2 have too few edges). Since the total remaining length is 24 and we have $9+5+5$ left to pack, the final cycle must be of length 5, contradicting the fact that there are at most two C_5s remaining.

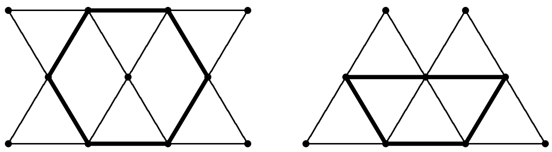


FIGURE 3. 2-balanced 6-cycle and 1-balanced 5-cycle of triangles.

Hence we can always pack a subset of cycles into a graph of the form $O^{\cdot n-1} \cdot L_m$ (or $O^{\cdot n}$). We now need to show that we can pack L_m and the remaining cycles (including C_3s and C_6s) into O when the total length is 12. If $m = 0$ pack one of the remaining cycles C_{m_i} as L_{m_i} first. Hence we may assume $m > 0$. If none of the remaining cycles are C_3, C_4, or C_6, then we are done by Theorem 2. We used all the C_4s at the beginning of the proof, so the only other combinations are those listed in Lemma 3. Hence in all cases we are done. □

3. EULERIAN TRAILS OF TRIANGLES

Let H be a graph with an edge-decomposition into triangles $\mathcal{T}$, so $E(H) = \bigcup_{T\in\mathcal{T}} E(T)$. Define a *trail* of triangles as a trail P (of edges) in H such that the edges of P lie in distinct triangles of $\mathcal{T}$. (See Figure 1 for an example where the trail is a path.) We shall refer to the triangles of $\mathcal{T}$ that contain an edge of P as the *triangles associated with* P. We shall call P the *underlying* trail when we wish to emphasize the trail rather than this set of triangles. Throughout this section, whenever we refer to a triangle it will always be assumed that it is a triangle in $\mathcal{T}$. Define a k*-balanced cycle* of triangles as a trail of triangles in which the underlying trail is a cycle and for any $v \in V(H)$ there are at most k triangles in $\mathcal{T}$ with one edge in the cycle and the opposite vertex equal to v (see Figure 3). In this section $n = |V(H)|$.

Lemma 5. *If $n = |V(H)|$, $\delta(H) \geq \frac{3}{4}n + k + \frac{n}{2k} + 3$, and H has a triangle decomposition $\mathcal{T}$, then H has a Hamiltonian k-balanced cycle of triangles (i.e., the underlying cycle is an n-cycle).*

Proof. First we show that H has some k-balanced cycle of triangles. Form a subgraph H' of H by taking one edge from each triangle in $\mathcal{T}$. Now $|E(H')| = \frac{1}{3}|E(H)| \geq \frac{n}{6}\delta(H)$. By the arithmetic-geometric mean inequality, $k + \frac{n}{2k} \geq \sqrt{2n}$. Since $\delta(H) \leq n-1$ it is easily checked that $k \geq 2$ and $\sqrt{2n} \leq \frac{n}{4}$. Hence

$$\delta(H) \geq \frac{3}{4}n + \sqrt{2n} + 3 \geq 4\sqrt{2n} + 3 > 3(\sqrt{n} + 1).$$

Thus $|E(H')| > \frac{1}{2}(n-1)\sqrt{n} + \frac{n}{2} \geq \mathrm{ext}(C_4, n)$, the extremal number of C_4 (see [4, p.310]). Thus H' contains a C_4. This C_4 is a 2-balanced cycle of triangles in H since each vertex v can be in triangles opposite at most 2 edges of this C_4. We shall now extend this balanced cycle of triangles.

Let x_1 and x_2 be adjacent vertices on a k-balanced cycle $C = x_1 \dots x_L$ of length L. We shall try to extend the underlying cycle C by replacing x_1x_2 by x_1vx_2 for some vertex v. There are at least $d_H(x_1) + d_H(x_2) - n \geq \frac{n}{2} + 2k + \frac{n}{k} + 6$ vertices adjacent to both x_1 and x_2. Of these, at most $L-2$ lie on C. The third vertex of the triangles of $\mathcal{T}$ containing x_1v are distinct for distinct v. Thus at most L/k edges x_1v from x_1 cannot be used to extend C since the third vertex in the triangle on this edge has already been used k times. Similarly at most L/k edges from x_2 cannot be used. Finally, the edges to the third vertices in the triangles on x_Lx_1, x_2x_3 and x_1x_2 are excluded, since then x_1v or x_2v would lie in a triangle that has already been used (or would lie in the same triangle). Hence, provided $\frac{n}{2}+2k+\frac{n}{k}+6>L+2L/k+1$ we can find a vertex v and extend the cycle by replacing the edge x_1x_2 with x_1vx_2. However, if $L\leq\frac{n}{2}+2k$ then $L+2L/k+1\leq\frac{n}{2}+2k+\frac{n}{k}+5$. Thus we can extend C at least until $L>\frac{n}{2}+2k$.

Now assume $L<n$ and pick a vertex $v\notin V(C)$. We shall try to extend the cycle further using this vertex. There are at least $d_H(v)+L-n$ vertices x_i in the underlying cycle C adjacent to v. Of these at most L/k edges vx_i cannot be used since the third vertex of the triangle on vx_i has been used k times. At most another $2k$ edges from v cannot be used, since they lie in triangles already associated with C. Now if $d_H(v)+L-n-L/k-2k>\frac{L}{2}$ then v is adjacent to a pair of adjacent vertices on C by "good" edges. In this case we can extend C as before by replacing x_ix_{i+1} by x_ivx_{i+1} where vx_i and vx_{i+1} are adjacent good edges. This inequality holds whenever $\frac{L}{2}-\frac{L}{k}>\frac{n}{4}+k-\frac{n}{2k}-3$. However, if $L>\frac{n}{2}+2k$ then $\frac{L}{2}-\frac{L}{k}>\frac{n}{4}+k-\frac{n}{2k}-2$. Hence we can extend C until $L=n$. Thus a Hamiltonian k-balanced cycle exists. □

Define an *Eulerian trail* of triangles as a closed trail of triangles which uses an edge from every triangle of $\mathcal{T}$. We call this trail *good* if the underlying trail meets every vertex of H.

Lemma 6. *If $\delta(H)\geq\frac{3}{4}n+\sqrt{6n}+10$ and H has a decomposition into triangles, then H contains a good Eulerian trail of triangles.*

Proof. Let $k=\lceil\sqrt{n/6}\,\rceil$. Then $\delta(H)\geq\frac{3}{4}n+3k+\frac{n}{2k}+7$. By Lemma 5, H contains a k-balanced Hamiltonian cycle of triangles C. Removing the triangles associated with C from H and applying Lemma 5 again we get a second k-balanced Hamiltonian cycle C'. (The minimum degree after removing the triangles associated with C is at least $\delta(H)-4-2k\geq\frac{3}{4}n+k+\frac{n}{2k}+3$.) Pick one edge out of every triangle in $\mathcal{T}$ so that for the triangles associated with C no edge of C is selected and for the triangles associated with C' only edges from C' are selected. For the remaining triangles in $\mathcal{T}$ pick the edges arbitrarily. Let H' be the graph with these edges. It is enough to show that we can choose the edges above so that H' is Eulerian. First assume H' has some vertices of odd degree. Let $C=v_1v_2\dots v_n$ and look at each v_i in turn. If v_i has odd degree, change the edge chosen in the triangle on v_iv_{i+1}. This triangle is $v_iv_{i+1}x$, say, and either edge xv_i or xv_{i+1} has been chosen. By changing the choice of edge we change the parity of the degree at v_i and v_{i+1} only. Repeating this process for each i in turn, we get a choice of edges which make the degrees at $v_1,\dots,v_{n-1}$ even. By degree sums, v_n must now also be even and we are done since all n vertices of H' are now of even degree. The choices of edges chosen from the triangles associated with C' have not been changed and so H' has a Hamiltonian cycle C'. Hence H' is connected and even, so Eulerian. An Eulerian trail of H' gives an Eulerian trail of triangles of H. Since it contains the edges of a Hamiltonian cycle, the underlying trail meets every vertex of H, and so the trail is good. □

Corollary 7. *If H has a decomposition into triangles and $\delta(H)\geq\frac{3}{4}n+\sqrt{6n}+10$, then Theorem 1 holds for $G=H[2]$.*

Proof. We now need to prove Theorem 1 under the assumption that $G=H[2]$ and H has a packing with a good Eulerian trail of triangles. Hence G has a packing with a closed trail of linked octahedra. These can be packed by any combination of closed trails by Theorem 4. Every packed cycle meets some link vertex of some octahedron, so since the Eulerian trail of triangles is good, we can start the packing at an appropriate point on the closed trail so that C_1 meets any particular vertex pair v_1, v_2. In any octahedron that contains them, v_1 and v_2 are symmetric. Thus we can change the packing of closed trails in the octahedra if necessary so that C_1 meets $v = v_1$, say. □

4. Graphs of even order

In this section we extend the result to all graphs of even order. For this it is necessary to assume that there are many closed trails of large lengths. Hence we shall also need to consider the cases when almost all the closed trails are short. For this we use a powerful result of Caro and Yuster [5, Theorem 4.1] on list packings. The following theorem is just a special case of this result.

Theorem 8. *For any positive integer L there exist $N(L)$ and $\epsilon(L) > 0$ such that for any even graph G on n vertices with $n \geq N(L)$ and $\delta(G) \geq (1 - \epsilon(L))n$ and for any collection of integers $m_1, \dots, m_t$ with $3 \leq m_i \leq L$ and $\sum_{i=1}^{t} m_i = |E(G)|$ one can write G as the edge-disjoint union of cycles $C_{m_1}, \dots, C_{m_t}$.*

This result is in turn derived from a result of Gustavsson [6]. It is worth noting that the value of $\epsilon(L)$ given is extremely small, in particular $\epsilon(3) = 10^{-24}$. We shall also need the following lemma.

Lemma 9. *Assume x and y are vertices of G (possibly equal) with $d_G(x) + d_G(y) \geq \frac{4n}{3}$, and assume also that $|E(G)| \geq m + (1 - \epsilon)\frac{n^2}{2}$ for some integer $m \geq 2$ ($m \geq 3$ if $x = y$) and some ϵ, $0 < \epsilon < \frac{1}{9}$. Then one can find a trail $P = x_0 x_1 \dots x_m$ of length m with $x_0 = x$, $x_m = y$ and such that $d_{G \setminus P}(x_i) \geq (1 - 3\epsilon)n$ for all i with $0 < i < m$.*

Proof. First assume $m = 2$. It is sufficient to find a vertex v with $d_G(v) \geq (1 - 3\epsilon)n + 2$ and $xv, yv \in E(G)$, since then xvy is a suitable trail. There are at least $d_G(x) + d_G(y) - n \geq \frac{n}{3}$ vertices adjacent to both x and y. If all of these vertices had degree less than $(1 - 3\epsilon)n + 2$, then these vertices would all have degree more than $3\epsilon n - 3$ in the complement of G. Hence the complement of G would contain more than $(3\epsilon n - 3)(\frac{n}{3})/2 = (\epsilon n - 1)\frac{n}{2}$ edges. Thus G would contain less than $\binom{n}{2} - (\epsilon n - 1)\frac{n}{2} = (1 - \epsilon)\frac{n^2}{2}$ edges, a contradiction. Therefore such a v must exist.

Now assume $m > 2$. As before, there are at least $\frac{n}{3}$ vertices adjacent to both x and y and at least one of these, v say, has degree at least $(1 - 3\epsilon)n + 2 > \frac{2n}{3} + 2$. Assume without loss of generality that $d_G(x) \geq d_G(y)$. Then $d_G(x) \geq \frac{2n}{3}$ and after removing edge vy, $d_{G \setminus \{vy\}}(x) + d_{G \setminus \{vy\}}(v) \geq \frac{4n}{3}$ (even if $x = y$). Now use induction to find a trail between v and x of length $m - 1$ in $G \setminus \{vy\}$. Adding the edge vy to this trail gives the required trail in G. The degree condition $d_{G \setminus P}(v) \geq (1 - 3\epsilon)n$ holds by induction if v occurs in the interior of the trail of length $m - 1$, otherwise $d_{G \setminus P}(v) = d_G(v) - 2 \geq (1 - 3\epsilon)n$. □

We shall also use the result that if $\delta(G) > \frac{n}{2}$ then G is pancyclic, i.e., contains cycles of all lengths from 3 to n (see for example [4, p.150]).

Lemma 10. *Theorem 1 holds for all graphs of even order.*

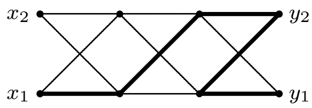


FIGURE 4. Trail of length 5 from x_1 to y_1 in $P_1[2]$.

Proof. Choose N and ϵ so that $N \geq \max(N(25), 2N(3), 10^3)$ and $\epsilon' = 2\epsilon + \frac{7}{N} \leq \epsilon(25)/94$, where $N(L)$ and $\epsilon(L)$ are the functions of Theorem 8. We can assume $\epsilon(25) \leq \min\left(\epsilon(3), \frac{1}{9}\right)$.

First we find a large subgraph of G of the form $H[2]$ with $G \setminus H[2]$ Eulerian and H decomposable into triangles. Since $\delta(G) \geq (1-\epsilon)n > \frac{n}{2}$, G contains a Hamiltonian cycle C. Pair up the vertices as $V(G) = \bigcup_{i=1}^{n/2}\{x_i, y_i\}$. Let H' be the maximal graph on the $\frac{n}{2}$ vertices $v_i = \{x_i, y_i\}$ such that $H'[2]$ is a subgraph of $G \setminus C$. Each edge not in H' corresponds to four edges, at least one of which is not in $G \setminus C$. Hence $d_{H'^c}(v_i) \leq d_{(G\setminus C)^c}(x_i) + d_{(G\setminus C)^c}(y_i)$. Thus $\Delta(H'^c) \leq 2\epsilon n + 2$, and so $\delta(H') \geq (1-4\epsilon)\frac{n}{2} - 3 > (1-2\epsilon')\frac{n}{2} > \frac{n}{4}$. Now H' has a Hamiltonian cycle $v_1 \dots v_{n/2}$. For each $i = 1, \dots, \frac{n}{2} - 1$ in turn, if v_i has odd degree, remove the edge $v_i v_{i+1}$ from H'. Thus, by removing some of the edges of this cycle we can find an even graph H'' with $E(H'') \subseteq E(H')$ and $\delta(H'') \geq (1-4\epsilon)\frac{n}{2} - 5$. Since $\delta(H'') > (1-2\epsilon')\frac{n}{2} > \frac{n}{4}$ we can also remove a C_4 or C_5 from H'' to get an even graph H_0 with $E(H_0)$ divisible by 3 and $\delta(H_0) \geq (1-4\epsilon)\frac{n}{2} - 7 \geq (1-2\epsilon')\frac{n}{2}$.

Since $G \setminus H_0[2]$ is even and connected (it contains the Hamiltonian cycle C), it is Eulerian. Let E_0 be an Eulerian trail of $G \setminus H_0[2]$ and let T_0 be a zero length subtrail of E_0 (i.e., a single vertex). Since $\delta(H_0) \geq (1-2\epsilon')\frac{n}{2}$, $\Delta(E_0) = \Delta(G \setminus H_0[2]) \leq 2\epsilon' n$. Hence $|E_0| \leq \epsilon' n^2$. Also

$$|E(H_0)| \geq (1-2\epsilon')\frac{n^2}{8} \geq (1-26\epsilon')\frac{n^2}{8} + 3|E_0|,$$

$$\delta(H_0) - 4\Delta(E_0) \geq (1-2\epsilon')\frac{n}{2} - 8\epsilon' n > (1-94\epsilon')\frac{n}{2}.$$

Our aim is to pack some closed trails so as to use up all the edges of E_0. In doing so, we may need to use some edges of $H_0[2]$, but we shall ensure that whenever we use edges from $H_0[2]$, the remaining graph is still of the form $H[2]$ with H even, of large minimum degree, and $|E(H)|$ divisible by 3. The purpose of T_0 (later T_i) is that it covers the vertices that are in danger of having their degrees in $H[2]$ reduced too much, and so should be removed when packing the next C_{m_i}.

Assume by induction that we have an even graph H_i on $\frac{n}{2}$ vertices with $|E(H_i)|$ divisible by 3 and a closed trail E_i in the complement of $H_i[2]$. Let T_i be a segment of this trail of length at most 21. Assume also that

$$|E(H_i)| \geq (1-26\epsilon')\frac{n^2}{8} + 3|E_i|, \tag{1}$$

and for all $v = \{v_1, v_2\} \in V(H)$,

$$d_{H_i}(v) - 2d_{E_i \setminus T_i}(v_1) - 2d_{E_i \setminus T_i}(v_2) \geq (1-94\epsilon')\frac{n}{2}. \tag{2}$$

Pick j_i with $m_{j_i} \geq 26$ and $|E_i| \geq m_{j_i} - 3$. Pick a subtrail of E_i of length $m_{j_1} - 5$ containing T_i. This is possible since $m_{j_i} - 5 \geq 21 \geq |T_i|$. Let x_1 and y_1 be the endvertices of this subtrail.

Assume x_1 lies in the vertex pair $\{x_1, x_2\}$ and y_1 lies in (possibly the same) vertex pair $\{y_1, y_2\}$. Join these vertex pairs with two trails P_1 and P_2 each of length 3 in H_i using Lemma 9. Let H_{i+1} be the graph H_i with these trails deleted. Note that H_{i+1} is even and $|E(H_{i+1})|$ is divisible by 3. In G, these trails correspond to two graphs $P_1[2]$ and $P_2[2]$, each made up of three C_4s as shown in Figure 4. There exists a trail of length 5 inside $P_1[2]$ joining x_1 and y_1. Combining this trail with the subtrail of length $m_{j_i} - 5$ above completes a packing of C_{j_i}. The remaining 7 edges of $P_1[2]$ form another trail from x_1 to y_1. The graph $P_2[2]$ is Eulerian and meets x_1, so combining these we get a trail of length 19 from x_1 to y_1 using the remaining edges of $P_1[2]$ and $P_2[2]$. Delete the subtrail of length $m_{j_i} - 5$ from E_i and add the trail of length 19 above to form a new closed trail E_{i+1}. Define T_{i+1} to be the trail of length 19 extended by one edge of E_{i+1} on either end (so that x_1 and y_1 are now interior points of T_{i+1}). Now $|T_{i+1}| = 21$ and since $m_i \geq 26$, $|E_{i+1}| \leq |E_i| - 2$. Condition (1) holds for $i+1$ since $|E(H_{i+1})| = |E(H_i)| - 6$.

We now check condition (2) with i replaced with $i+1$. Since $3|E_i| \geq 6$, we can take $\epsilon = 26\epsilon' < \frac{1}{9}$ in Lemma 9. Thus if v is in the interior of P_1 or P_2 then $d_{H_{i+1}}(v) \geq (1 - 78\epsilon')\frac{n}{2}$. But $E_{i+1} \setminus T_{i+1} \subseteq E_i \setminus T_i \subseteq \cdots \subseteq E_0$ and $\Delta(E_0) \leq 2\epsilon' n$. Hence $2d_{E_{i+1} \setminus T_{i+1}}(v_1) + 2d_{E_{i+1} \setminus T_{i+1}}(v_2) \leq 8\epsilon' n$ and condition (2) holds for v. The degree $d_{H_i}(v)$ has been reduced by 2 at each endpoint of these trails (or by 4 if the endpoints are the same). However $2d_{E_i \setminus T_i}(v_1) + 2d_{E_i \setminus T_i}(v_2)$ has also been reduced by at least 2 (or 4) since x_1 and y_1 lie in the interior of T_{i+1}. Hence condition (2) holds here. At all other vertices $d_{H_{i+1}}(v) = d_{H_i}(v)$ and $E_{i+1} \setminus T_{i+1} \subseteq E_i \setminus T_i$. Therefore (2) holds at all vertices.

We can now inductively construct E_i, T_i and H_i for $i > 0$. This process terminates when one of the following conditions occur

(1) $|E_i| < m_{j_i} - 3$; or
(2) no m_js are left with $m_j \geq 26$.

In the first case, the next closed trail can be split as E_i and another closed trail of length $m = m_{j_i} - |E_i| > 3$ that meets some vertex v of E_i. Pack this closed trail of length m as C_1 and all remaining C_js into $H_i[2]$ using Corollary 7. We use Theorem 8 to pack H_i with triangles. For this to succeed, we need

$$\delta(H_i) \geq \frac{3n}{8} + \sqrt{3n} + 10, \qquad \delta(H_i) \geq (1 - \epsilon(3))\frac{n}{2}, \qquad \frac{n}{2} \geq N(3). \tag{3}$$

In the second case, we are left with all the closed trails of length ≤ 25 to be packed into the graph $H_i[2] \cup E_i$. Once again, Theorem 8 will provide this packing provided

$$\delta(H_i[2] \cup E_i) \geq 2\delta(H_i) \geq (1 - \epsilon(25))n, \qquad n \geq N(25). \tag{4}$$

Since $n \geq N \geq 10^3$, $\frac{3n}{8} + \sqrt{3n} + 10 \leq (1 - \frac{1}{9})\frac{n}{2}$. However, $\delta(H_i) \geq (1 - 94\epsilon')\frac{n}{2}$, $94\epsilon' \leq \epsilon(25) \leq \min\left(\epsilon(3), \frac{1}{9}\right)$ and $N \geq 2N(3), N(25)$. Hence conditions (3) and (4) hold and we are done. □

5. Graphs of odd order

Lemma 11. *Theorem 1 holds for all graphs of odd order.*

Proof. We know Theorem 1 is true for even n with $N = N_1$ and $\frac{1}{3} > \epsilon = \epsilon_1 > 0$, say. Assume n is odd and set $\delta_0 = (1-\epsilon_1)(n-1)$. Let G be an even graph of odd order $n \geq \max(N_1, \frac{60}{\epsilon_1})$ and $\delta(G) \geq (1-\frac{\epsilon_1}{4})n$. Let X be the neighborhood of $v \in V(G)$. Now $\delta(G[X]) \geq (1-\frac{\epsilon_1}{2})n \geq |X|/2$ so $G[X]$ contains a Hamiltonian cycle. Since $|X| = d_G(v)$ is even, by taking every other edge in this cycle we get a 1-factor I of $G[X]$. Joining this 1-factor to v we get a set of triangles with common vertex v. Let $G' = (G - v) \setminus I$ and note that $\delta(G') \geq \delta(G) - 2 \geq \delta_0$. Now pack closed trails C_i of length m_i into these triangles starting with C_1 (and so ensuring that C_1 meets v). If m_i is divisible by 3 then we pack an exact number of triangles. Otherwise we can pack most of this closed trail and are left with length 4 or 5 still to pack. For this, add back an unused edge xy of I to G' and then attach a trail of length 2 or 3 between x and y in G'. This trail together with xv and yv allows us to pack the remaining part of C_i. We then remove these edges from G' and repeat with the next C_i until all edges from v have been used. We now show that we can do this by Lemma 9 while keeping $\delta(G') \geq \delta_0$. We added back the edge xy, so $d(x), d(y) > \delta_0$. Also $|E(G')| \geq |E(G)| - \frac{5n}{2}$, since at worst we have removed a C_5 for every edge in I. Hence at each stage

$$|E(G')| \geq \left(1 - \frac{\epsilon_1}{4}\right)\frac{n^2}{2} - \frac{5n}{2} \geq \left(1 - \frac{\epsilon_1}{3}\right)\frac{(n-1)^2}{2} + 3$$

since $\epsilon_1 n \geq 60$ and $\epsilon_1 < \frac{1}{3}$. Hence we can apply Lemma 9 as claimed keeping $\delta(G') \geq \delta_0$. If we repeat until all the edges from v are used, part of the last closed trail C_i to be packed may be unused. If this is the case, the total length remaining to be packed will be at least 3, so we include a closed trail of this length as our C_1 when applying Theorem 1 to G' and set v to be any vertex of G' that the part of C_i already packed meets. Since we have already packed all the edges of $G \setminus G'$, Theorem 1 holds for G. □

Theorem 1 now follows from the previous two lemmas.

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