Decompositions of graphs into cycles with chords

Paul Balister*† Hao Li‡ Richard Schelp*

May 22, 2017

In memory of Dick Schelp, who passed away shortly after the submission of this paper.

Abstract

We show that if G is a graph on at least $3r + 4s$ vertices with minimum degree at least $2r + 3s$, then G contains $r + s$ vertex disjoint cycles, where each of s of these cycles either contain two chords, or are of order 4 and contain one chord.

Keywords: graphs, cycles, decompositions.

1 Introduction and Main Result

The following beautiful conjecture of Bialostochi, Finkel, and Gyárfás appears in [1].

*Department of Mathematical Sciences, University of Memphis, TN 38152, USA. E-mail: pbalistr@memphis.edu.

†Partially supported by NSF grants CCF-0728928 and DMS-1600742.

‡Laboratoire de Recherche en Informatique UMR 6823 Université Paris-sud 11 and CNRS Orsay F-91405, France. E-mail: li@lri.fr.

Conjecture 1. *Let r, s be nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and minimum degree $\delta(G) \geq 2r + 3s$. Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.*

The complete bipartite graph $K_{2r+3s-1,n-2r-3s+1}$ shows that the minimum degree cannot be lowered when $n \geq 4r + 6s - 2$.

The conjecture is a generalization of well known results of Pósa, and of Corrádi and Hajnal. Pósa proved (see [7, problem 10.2]) that any graph with minimum degree at least 3 contains a chorded cycle and Corrádi and Hajnal [3] proved that any graph of minimum degree at least $2r$ of order $n \geq 3r$ contains r vertex disjoint cycles.

The purpose of this article is to show that a stronger result than that given in the conjecture is true. We prove the following theorem.

Theorem 2. *If G is a simple graph on $|V(G)| \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ vertex disjoint cycles, each of s of them either with two chords, or a C_4 with one chord.*

It is likely the case that among the chorded "long" cycles more than two chords will be present, and that one can insist on two chords even in the C_4s, but our method of proof does not establish this.

It has come to our attention that Conjecture 1 has been proved by Gao, Li, and Yan and appears in [4], but they do not address the stronger result given in our theorem. Also a degree sum condition is used by Chiba, Fujita, Gao, and Li in [2], and neighborhood union conditions are used by Gao, Li, and Yan [5], and by Qiao [8], to realize disjoint chorded cycles. Finally, independently of our results, Gould, Hirohata, and Horn [6] proved a result on the existence of disjoint doubly chorded cycles under a degree sum condition. This result however only applies to the $r = 0$ case with $|V(G)| \geq 6s$.

The proof of our theorem is based on several technical theorems and lemmas, the last two of which are in themselves of special interest. One of these (Theorem 12) generalizes the result of Pósa mentioned earlier by showing that a graph with minimum degree 3 contains a cycle with two chords.

We write as usual P_n, C_n, K_n, or E_n for a path, cycle, complete graph, or empty graph respectively on n vertices. When the number of vertices is unspecified we shall write for example P_* or C_*. We denote by C_n^{+k} any cycle of length n with at least k additional chords, and K_n^{-k} the complete graph on n vertices with at most k edges removed. If $k = 1$ then we write just C_n^+ or K_n^- for brevity. Note that, for example, a C_n^{+3} graph is also considered as a special case of a C_n^{+2} graph. It will also be convenient to denote by $C_*^\dagger$ a graph that is *either* a C_*^{+2} *or* a C_4^+. We shall write $G \cup H$ for the vertex disjoint union of G and H, and $G \cup H + ke$ for such a graph with k additional edges added *between* G and H. We shall also use the notation $H \subseteq G$ or $G \supseteq H$ to indicate that H is a subgraph of G, and $H \subset G$ or $G \supset H$ to indicate that H is a non-spanning subgraph of G, i.e., a subgraph with $|V(H)| < |V(G)|$. For example, the statement $G \supset C_*$ indicates that G contains a non-hamiltonian cycle. We shall occasionally abuse notation by regarding a subgraph $H \subseteq G$ also as a subset of vertices of G. So for example, we write $G[H \cup v]$ instead of $G[V(H) \cup \{v\}]$ for the subgraph induced by the vertices of H and an extra vertex $v \in V(G)$.

The proof of Theorem 2 involves a number of technical theorems and lemmas, the relevance of which only becomes apparent in the proof of Theorem 2. Thus we shall first give the proof of Theorem 2 assuming these results, and then state and prove them in the next section.

Proof of Theorem 2. Consider all possible decompositions of the graph G into $r + s$ vertex disjoint subgraphs G_i, $i = 1, \ldots, r + s$, each subgraph being of one of the following types

$$C_*, \quad C_*^{+2}, \quad C_4^+, \quad K_4, \quad K_5, \quad K_6^-, \quad K_7^{-3},$$

and possibly an additional set S of unused vertices. By [3] there is a collection of $r+s$ disjoint cycles in G, so at least one such decomposition exists. Among these decompositions, pick one with the minimal number of C_*s. Say there are r' C_*s and $s' = r+s-r'$ of the other subgraphs. If $r' \le r$ then we are done, as each of the other graphs on this list contains (and hence can be replaced with) a cycle while K_4, K_5, K_6^- and K_7^{-3} all contain a C_4^+ subgraph. Hence we may assume $r' > r$. Among the decompositions with this minimal r', we shall take one of minimal *weight*, where the weight of a decomposition is defined as the sum of certain weights assigned to each of the subgraphs G_i. The weights $w(G_i)$ assigned to these subgraphs are given in Table 1. Here ε is chosen so that $0 < \varepsilon < \frac{1}{7}$ and we regard a subgraph as a C_* only if it fails to have enough chords to be a $C_*^\dagger$. We call such a decomposition with

Table 1: Weights of graphs ($0 < \varepsilon < \frac{1}{7}$).

G_i	$w(G_i)$
$C_n, \quad n \geq 3$	n
$C_n^{+2}, n \geq 5$	n
C_4^+	4
K_4	$4 - 4\varepsilon$
K_5	$4 - 5\varepsilon$
K_6^-	$4 - 6\varepsilon$
K_7^{-3}	$4 - 7\varepsilon$

this r' and minimal weight an *optimal* decomposition. Note that as we are assuming for contradiction that $r' > r$, there will always be at least one C_* in our optimal decomposition.

Claim 1: $S = \emptyset$ in any optimal decomposition.

Proof of Claim 1. Suppose otherwise and pick $v \in S$. By Lemma 4 below, v can send at most 3 edges to any $G_i \neq C_*$, otherwise we could construct a new decomposition replacing G_i with a $G_i' \neq C_*$ of smaller weight, or two C_4^+s, which on discarding a C_* from our decomposition would result in a decomposition with smaller r'. Similarly, by Lemma 3, v can send at most 2 edges to any $G_i = C_*$. Let G_1 be one of the C_* cycles. Then $d(v) \leq 2(r'-1) + 3s' + d_{S \cup G_1}(v)$ where $d_{S \cup G_1}(v)$ is the degree of v in the subgraph $G[S \cup G_1]$. But $d(v) \geq 2r + 3s \geq 2r' + 3s' + 1$. Thus $d_{S \cup G_1}(v) \geq 3$ for every $v \in S$. But then by Theorem 13, $G[S \cup G_1]$ contains either a cycle of smaller order than G_1, or a C_*^{+2}, either of which could be swapped with G_1 to obtain a better decomposition. Thus $S = \emptyset$. □

Claim 2: There are no C_ns with $n > 3$ and no C_n^{+2}s with $n > 4$ in any optimal decomposition.

Proof of Claim 2. Consider a subgraph G_i with maximal weight in an optimal decomposition of G. Suppose $G_i = C_n$ with $n \geq 4$. The sum of the degrees $d_{G_i}(v)$ over $v \in G_i$ is at most $2n + 2$ as otherwise G_i would have two chords. If G_i sends $2n + 1$ edges to any $G_j = C_*$, $j \neq i$, then by Lemma 10 we can replace G_i and G_j in the decomposition with two disjoint C_*s with smaller total weight (when $n = 4$) or a C_* and a $C_*^\dagger$ (when $n \geq 5$) giving a decomposition with smaller r'. Note that we are assuming G_i has maximal weight so that $|V(G_j)| \leq |V(G_i)|$. Again by Lemma 10, if G_i sends $3n+1$ ($\geq 2n+1$ for $n \geq 5$ or ≥ 11 for $n = 4$) edges to any $G_j = C_*^\dagger$ (including $K_4 = C_4^{+2}$ as a special case), then we can replace $G_i \cup G_j$ with a $C_* \cup C_*^\dagger$ with fewer total number of vertices and hence smaller total weight. If G_i sends $3n+1 \geq 13$ edges to a $G_j \in \{K_5, K_6^-\}$, then there must be some vertex $v \in G_j$ that sends at least 3 edges to G_i. Then by Lemma 3, $G[G_i \cup v]$ contains a shorter cycle C than G_i, while $G_j - v$ contains a K_4, which has weight at most 2ε more than G_j. Replacing G_i and G_j in the decomposition by C and this K_4 gives a decomposition with at least $1 - 2\varepsilon$ smaller weight. Now suppose G_i sends $3n+1$ edges to $G_j = K_7^{-3}$. Then at least one vertex $v \in G_i$ sends at least 4 edges to K_7^{-3}, so by Lemma 4 we can decompose $G[v \cup G_j]$ into two C_4^+s. Replacing G_i and G_j by these gives a decomposition with smaller r'. Since $S = \emptyset$, and combining the above bounds, the sum of the degrees $d(v)$ in G of the vertices of G_i is at most $(2n+2) + 2n(r'-1) + 3ns' < (2r+3s)n$, so at least one vertex of G_i violates the minimum degree condition.

Now suppose the subgraph with maximal weight is $G_i = C_n^{+2}$ for some $n > 4$. As $C_5^{+2} \supset C_4^+$ and $w(C_5^{+2}) > w(C_4^+)$ we may assume $n \geq 6$. The sum of the degrees $d_{G_i}(v)$ over $v \in G_i$ is at most $3n$ as otherwise G_i would have more than $n/2$ chords and by Lemma 5 we could replace G_i by a subgraph of G_i of smaller weight. If G_i sends $2n+1$ edges to any $G_j = C_*$, then by Lemma 10 we can replace $G_i \cup G_j$ in the decomposition with a $C_* \cup C_*^\dagger$ with fewer total number of vertices and hence smaller weight. Note that $|V(G_j)| \leq |V(G_i)|$ as G_i has maximal weight. Similarly, if G_i sends $3n+1$ edges to any $G_j = C_*^\dagger$ (including $K_4 = C_4^{+2}$) then, by Lemma 11, we can replace $G_i \cup G_j$ with a $C_*^\dagger \cup C_*^\dagger$ with fewer total number of vertices and hence smaller weight. If G_i sends at least $3n+1 \geq 19$ edges to a $G_j \in \{K_5, K_6^-\}$, then there must be some vertex $v \in G_j$ that sends at least 4 edges to G_i. Then by Lemma 4, $G[G_i \cup v]$ contains a $C_*^\dagger$ on fewer vertices than G_i, while $G_j - v$ contains a K_4, which has weight at most 2ε more than G_j. Replacing G_i and G_j in the decomposition by this $C_*^\dagger$ and K_4 gives a decomposition with at least $1 - 2\varepsilon$ smaller weight. Now suppose G_i sends at least $3n+1$ edges to $G_j = K_7^{-3}$. Then some vertex $v \in G_i$ sends at least 4 edges to K_7^{-3} so by Lemma 4 we can decompose $G[v \cup G_j]$ into two C_4^+s. Replacing $G_i \cup G_j$ by these gives a decomposition with smaller weight (8 versus $n + 4 - 7\varepsilon \geq 10 - 7\varepsilon > 9$). Combining the above bounds and using $S = \emptyset$, the sum of the degrees in G of the vertices of G_i is at most $2nr' + 3ns' < (2r+3s)n$, so at least one vertex of G_i violates the minimum degree condition. $\square$

By Claims 1 and 2, the optimal decomposition consists only of graphs in the set $\{C_3, C_4^+, K_4, K_5, K_6^-, K_7^{-3}\}$ and the G_is cover all vertices of G. As $|V(G)| \geq 3r + 4s$ and $r' > r$, we must have at least one subgraph $G_i \in \{K_5, K_6^-, K_7^{-3}\}$. We now construct an "almost optimal" decomposition with a set S of unused vertices with $1 \leq |S| \leq 3$ and with no $|V(G_i)| > 4$ unless $|S| = 3$. If the optimal decomposition contains K_7^{-3}, then remove three vertices from it to produce a K_4. The S will then consist of the three removed vertices. If there is no K_7^{-3} then remove one vertex from a K_6^- to produce a K_5, or one from a K_5 to produce a K_4. Repeating this process at most three times and placing the removed vertices in S, we obtain a decomposition with a set S of unused vertices with either $|S| = 3$, or with $|S| < 3$ and no $|V(G_i)| > 4$. Note that although this decomposition no longer has minimal weight, its weight is in all cases at most 3ε larger than the optimal decomposition, and it has the same value of r'. As $r' > 0$ we also have, say, $G_1 = C_3$. We count the sum of the degrees of the vertices in $S \cup G_1$ with the vertices in S weighted by a factor of $3/|S|$, i.e., we estimate

$$E = \frac{3}{|S|} \sum_{v \in S} d(v) + \sum_{v \in V(G_1)} d(v).$$

Suppose $G_j = C_3$, $j > 1$. Each $v \in S$ can send only one edge to G_j, otherwise $G[v \cup G_j]$ would contain a C_4^+ that we could swap with G_j giving a decomposition with smaller r' (and hence a smaller r' than the optimal decomposition). Clearly $G_1 = C_3$ can send at most 9 edges to $G_j = C_3$, so we have a contribution of at most $3 + 9 = 12$ to E from edges from $S \cup G_1$ to this G_j.

Now suppose $G_j \in \{C_4^+, K_4\}$ and suppose further that there is a contribution to E of more than 18 from edges between $S \cup G_1$ and G_j. Clearly there are at most 12 edges between $G_1 = C_3$ and G_j, so there must be a contribution of more than 6 to E from the edges between S and G_j. Thus there exists $v \in S$ which sends more than 2 edges to G_j. If $G_j = C_4^+$ and $G[v \cup G_j] \supseteq K_4$ then we can swap G_j with this K_4, decreasing the weight by 4ε. This would give a better decomposition than the optimal one, a contradiction. Hence we may assume v is joined to both degree 2 vertices and one of the degree 3 vertices in G_j (see Figure 1). Any single vertex u of G_j can be removed

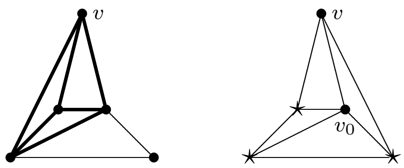

Figure 1: If $H = \{v\} \cup C_4^+ + 3e \not\supseteq K_4$ then for any $u \in C_4^+ - v_0$, $H - u \supseteq C_4^+$.

from $G[v \cup G_j]$ yielding a C_4^+ except possibly for v_0, a degree 3 vertex of G_j that is joined to v. If $G_j = K_4$ then any vertex of $G[v \cup G_j]$ can be removed yielding a C_4^+ or K_4. Thus if there are more than $3 + 1 + 1 + 1 = 6$ edges from G_j to G_1 we can find a $u \in G_j - v_0$ joined to two vertices of G_1 forming a C_4^+ on $G_1 \cup u$ and a C_4^+ on $v \cup G_j - u$. Replacing $G_1 \cup G_j$ with these two C_4^+s gives a decomposition with smaller r', so we may assume that there are at most 6 edges from G_1 to G_j. But there at most 4 edges from each $v \in S$ to G_j. Thus we have a contribution of at most $3 \times 4 + 6 = 18$ to E from the edges to G_j.

Now suppose $G_j \in \{K_5, K_6^-, K_7^{-3}\}$. Then $|S| = 3$. By Lemma 6 there can be at most $12 + 5 = 17$ edges from $S \cup G_1$ to a K_5, otherwise we could find two C_4^+s in $G[S \cup G_1 \cup G_j]$ which could be swapped with $G_1 \cup G_j$ giving a decomposition with smaller r'. Similarly there are at most $12 + 6 = 18$ edges from $S \cup G_1$ to a K_6^- and at most $9 + 7 = 16$ edges from $S \cup G_1$ to a K_7^-. Hence there is a contribution of at most 18 to E from edges to this G_j.

Each $v \in S$ can send at most one edge to G_1, otherwise we would have a C_4^+ which could be swapped with G_1, and each $v \in S$ can send at most $|S| - 1 \leq 2$ edges to S. Thus the contribution to E from edges within $G[S \cup G_1]$ is at most 6 from edges in G_1, plus 6 from edges between G_1 and S, and 6 from edges in S. Equality occurs only if $|S| = 3$ and $G[S \cup G_1]$ is two triangles with three edges joining them. But in this case either $G[S \cup G_1] \supseteq C_6^{+2}$ (if the three edges form a matching) or $G[S \cup G_1] \supseteq C_4^+$ (otherwise). This G_6^{+2} or C_4^+ can be swapped with G_1 reducing r', a contradiction. Hence the contribution to E from edges in $G[S \cup G_1]$ is at most 17.

In total, we have $E \leq 12(r' - 1) + 18s' + 17 < 6(2r + 3s)$ contradicting the fact that $E \geq 6\delta(G) \geq 6(2r + 3s)$. Thus a decomposition exists with $r' = r$ and the theorem is proved. □

2 Technical lemmas

Lemma 3. *If* $G = E_1 \cup C_n + 3e$ *then either* $G \supseteq C_m$ *for some* $m < n$ *or* $n = 3$ *and* $G = K_4$.

Proof. Let $E_1 = \{v\}$. If v sends three edges to C_n then the shortest arc between neighbors of v on C_n is of edge length at most $n/3$ and hence there is a cycle through v of length at most $n/3 + 2$. If $n > 3$ then we are done as $n/3 + 2 < n$. If $n = 3$ then $G = K_4$. □

Lemma 4. *For* $n > 4$, $E_1 \cup C_n^{+2} + 4e \supset C_m^{\dagger}$ *for some* $m < n$. *Also* $E_1 \cup C_4^{+} + 4e \supset K_4$, $E_1 \cup K_4 + 4e = K_5$, $E_1 \cup K_5 + 4e = K_6^{-}$, $E_1 \cup K_6^{-} + 4e = K_7^{-3}$, *and* $E_1 \cup K_7^{-3} + 4e \supseteq C_4^{+} \cup C_4^{+}$.

Proof. Suppose first that $H = C_n^{+2}$ with $n > 4$ and write $E_1 = \{v\}$. Let vv_i, $i = 1, \ldots, 4$, be the edges from v to H with v_i ordered cyclically around the main cycle C of H. If the arc between any consecutive v_i on C, say v_4 and v_1, contains at least two interior vertices, then $vv_1 \ldots v_2 \ldots v_3 \ldots v_4 v$ is a shorter cycle with two chords vv_2, vv_3 (Figure 2(a)). Thus we may assume there is at most one vertex of C between each consecutive pair of v_i, say $C = v_1u_1v_2u_2v_3u_3v_4u_4v_1$ where some of the u_i may not exist. If three consecutive v_i exist with no u_i between them then they form a C_4^{+} with v (Figure 2(b)). Hence we may assume that no two consecutive u_i are both missing.

Now consider the chords of H. If one of these chords joins a pair of consecutive v_is, say v_1v_2 then $\{v, v_1, v_2, u_1\}$ induces a C_4^{+} (Figure 2(c)). If one of the chords joins opposite v_i, say v_1v_3, then $vv_1u_1v_2u_2v_3v$ is cycle with two chords v_1v_3 and vv_2 (Figure 2(d)). Note that either u_3 or u_4 exists, so this cycle has length less than n. If a chord joins a u_i to a v_j, say u_1 to v_3 then $vv_1u_1v_3u_2v_2v$ is a cycle with chords u_1v_2 and vv_3 (Figure 2(e)). Once again, either u_3 or u_4 exists so this cycle has length less than n. Thus we may assume both chords are between u_i vertices. If two consecutive u_i are joined by a chord, say u_1u_2, then $vv_1u_1v_2u_2v_3v$ is a shorter cycle with chords vv_2 and u_1u_2 (Figure 2(f)). The only remaining case is when *both* u_1u_3 and u_2u_4 are chords. But then $vv_1u_1u_3v_3u_2v_2v$ is a shorter cycle with two chords u_1v_2 and vv_3 (Figure 2(g)).

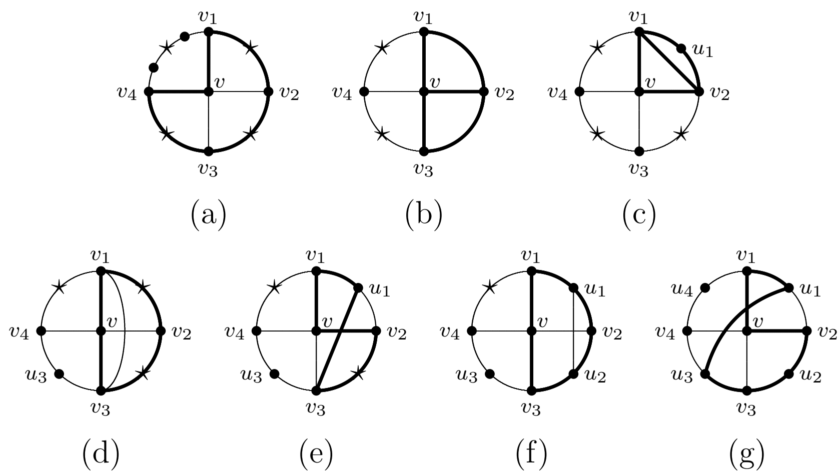

Figure 2: Finding H' in Lemma 4. Stars indicate possible presence of vertices.

If $H = C_4^+$ then $E_1 \cup H + 4e$ contains a K_4 (see Figure 1). Also it is clear that $E_1 \cup K_4 + 4e = K_5$, $E_1 \cup K_5 + 4e = K_6^-$, and $E_1 \cup K_6^- + 4e = K_7^{-3}$. Finally, suppose $H = K_7^{-3}$. As H is missing at most three edges and v is joined to 4 vertices of H, there must be a triangle vv_1v_2 in $G = E_1 \cup H + 4e$. Let $H' = H - \{v_1, v_2\}$. Then $H' = K_5^{-3}$. If H' is missing exactly 3 edges, one can find a vertex $u \in H'$ such that $H' - u \supseteq C_4^+$ (pick u to be a vertex of degree at least two in the complement of H'). Similarly, if H' is missing exactly 2 edges then there are at least three choices for u so that $H' - u \supseteq C_4^+$ (any vertex incident to a missing edge). For $H' = K_5^-$, $H' - u$ always contains a C_4^+. Thus in general, if x edges are missing from H' there are at most $x+1$ vertices u such that $H' - u \not\supseteq C_4^+$. There are also at most $3-x$ missing edges between H' and $\{v_1, v_2\}$ in H, so at most $3-x$ values of u such that uv_1v_2v is not a C_4^+. As $(3-x)+(1+x) < 5$, there is a u such that both uv_1v_2v is a C_4^+ and $H' - u \supseteq C_4^+$ so $G \supseteq C_4^+ \cup C_4^+$. □

Suppose a cycle C has two chords $e_1 = u_1v_1$ and $e_2 = u_2v_2$. We say that e_1 and e_2 *cross* if u_1, u_2, v_1, v_2 are all distinct and occur in the cyclic order u_1, u_2, v_1, v_2 on C. Note that incident chords are not considered crossing.

Lemma 5. *For $n \geq 8$, $C_n^{+5} \supset C_*^{+2}$. Also $C_7^{+3} \supset C_*^{+2}$, $C_6^{+4} \supset C_*^{+2}$, and $C_5^{+2} \supset C_4^+$.*

Proof. We prove the first statement, i.e., that $C_n^{+5} \supset C_m^{+2}$ for some $m < n$. The remaining low order cases can be checked by a case-by-case analysis (see [9]). Suppose we are given $C = C_n$ with five chords. If e is one chord and two other chords fail to cross e, then we obtain a smaller C_*^{+2} by simply shortening the cycle using one of the chords. Let e_1 and e_2 be two chords, chosen to be non-crossing if possible. Then the remaining three chords e_3, e_4, e_5 all cross both e_1 and e_2. Suppose e_3 and e_4 cross. Then using e_3, e_4 and two arcs of C we obtain a cycle with two chords e_1, e_2. This gives us our C_*^{+2} unless this cycle is hamiltonian, i.e., unless e_3 and e_4 are adjacent at both ends. A similar argument applies to e_4, e_5 and e_3, e_5. But we know at least two of these three pairs are crossing, otherwise one of e_3, e_4, e_5 would fail to cross two others. But if say e_3, e_4 and e_4, e_5 are crossing and adjacent at both ends with $e_3 \neq e_5$, then e_3, e_5 are crossing and not adjacent at both ends. Hence we always obtain a smaller C_*^{+2}. □

Lemma 6. *Each of the following graphs G satisfy $G \supseteq C_4^+ \cup C_4^+$.*

$$\begin{array}{lll} C_3 \cup K_5 + 6e & C_3 \cup K_6^- + 7e & C_3 \cup K_7^{-3} + 8e \\ E_3 \cup K_5 + 13e & E_3 \cup K_6^- + 13e & E_3 \cup K_7^{-3} + 10e \end{array}$$

Proof. If there are $|H| + 1$ edges between a C_3 and $H \in \{K_5, K_6^-, K_7^{-3}\}$ then there is a vertex $v \in H$ joined to two vertices of C_3 forming a C_4^+. But $H - v$ always contains a C_4^+.

Now suppose there are 13 edges between an E_3 and a K_5 (so only 2 of the possible edges are missing). One of the vertices $v \in E_3$ must send 5 edges to the K_5. Then $G[(E_3 - v) \cup K_5] = K_7^{-3}$ and v sends more than 4 edges to this graph, so we are done by Lemma 4.

Suppose there are 13 edges between an E_3 and a K_6^-. One of the vertices $v \in E_3$ must send at least 4 edges to the K_6^- forming a K_7^{-3} by Lemma 4. We then have at least $13 - 6 = 7$ edges between the two remaining vertices of E_3 and this K_7^{-3}. One of these vertices must send at least 4 edges to the K_7^{-3} forming two C_4^+s by Lemma 4.

Finally, suppose there are 10 edges between E_3 and K_7^{-3}. Then one of the vertices of E_3 sends at least 4 edges to the K_7^{-3} forming two C_4^+s by Lemma 4. □

Lemma 7. $C_* \cup C_* + 4e \supseteq C_*^{+2}$.

Proof. Assume the edges join $v_i \in C$ to $v_i' \in C'$, $i = 1, 2, 3, 4$, with the v_i and v_i' not necessarily distinct. If $v_1 = v_2 = v_3 = v_4$ then the v_i' are distinct and there is an arc, say $P = v_1' \dots v_2'$ of C' meeting v_3', v_4'. Then $v_1 P v_1$ is a cycle with chords $v_3 v_3'$, $v_4 v_4'$. Similarly we are done if all v_i' are equal. If $v_1 \neq v_2 = v_3 = v_4$ then v_2', v_3', v_4' are distinct, so v_1' is distinct from at least two other v_i'. Of course v_1 is also distinct from at least two other v_i as well. If no three of the v_i or v_i' are equal then it is automatically the case that v_1 and v_1' are distinct from at least two of the other v_i or v_i' respectively. Hence we may assume this. Then at least two v_i, $i > 1$, are such that there is an arc of C from v_1 to $v_i \neq v_1$ meeting all v_j (possibly as an end-vertex). Similarly for C'. Thus there is an $i > 1$ such that this holds for both v_i and v_i' with arcs P and P' respectively. The cycle $v_1 P v_i v_i' P' v_1' v_1$ has two chords $v_k v_k'$, $k \in \{2, 3, 4\} \setminus \{i\}$. □

Lemma 8. $P_* \cup C_* + 5e \supseteq C_*^{+2}$.

Proof. Fix an orientation of P and denote the edges between $P = P_*$ and $C = C_*$ as $e_i = u_i v_i$, $i = 1, \dots, 5$, where $u_i \in P$ are ordered by their location along P. (If two edges are incident to the same vertex on P we order them arbitrarily.) Suppose first that $v_1 \neq v_5$. Then one of the arcs in C from v_1 to v_5 in C must meet two of the three remaining v_i. Then the cycle $u_1 v_1 \dots v_5 u_5 \dots u_1$ contains at least two chords. Thus we may assume $v_1 = v_5$. Repeating this argument with v_4 in place of v_5 we are also done unless $v_4 = v_1$ or the cyclic ordering of the vertices on C is $v_1 = v_5, v_2, v_4, v_3$ with v_1, v_2, v_3, v_4 distinct. Similarly, using v_2 in place of v_1 we are done unless $v_2 = v_5$ or the cyclic ordering of the vertices on C is $v_1 = v_5, v_4, v_2, v_3$ and v_1, v_2, v_3, v_4 are distinct. Since these cyclic orderings are incompatible, we may assume either $v_4 = v_1$ or $v_2 = v_5$, say $v_4 = v_1$. But then v_1, v_2, v_3, v_4 are not distinct, so $v_2 = v_5$ also, and so $v_1 = v_2 = v_4 = v_5$. But then the cycle with edges e_1 and e_5 contains chords e_2 and e_4. □

Lemma 9. *$P_n \cup P_* + k_n e \supseteq C_*^\dagger$ and $P_n \cup P_* + k_n' e \supset C_*^\dagger$ where k_n and k_n' are given by the following table.*

n	1	2	3	4	5	≥ 6
k_n	4	5	6	6	7	8
k_n'	5	6	6	7	7	8

In particular $P_n \cup P_* + (n+4)e \supset C_*^\dagger$.

Proof. We first show that $P_* \cup P_* + 8e \supseteq C_*^{+2}$. Let the 8 edges between the paths be $e_i = u_i v_i$ with u_i on the first path P and v_i on the second path P'.

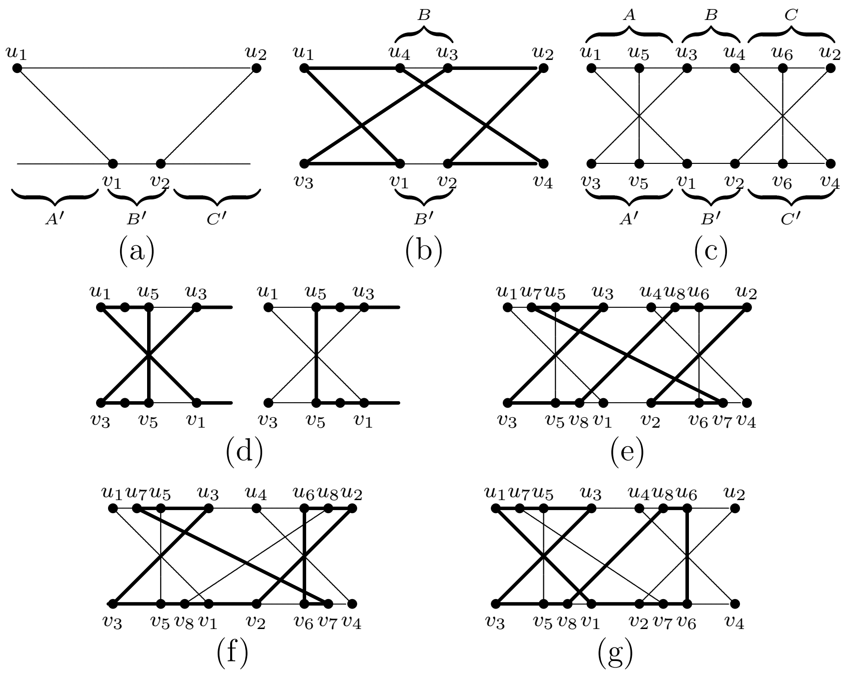

Figure 3: Finding a C_*^{+2} in $P_* \cup P_* + 8e$ in Lemma 9

Let e_1 and e_2 be the edges whose end-vertices u_1 and u_2 are furthest apart on P. The end-vertices v_1 and v_2 in general cut the other path P' into three pieces A', B', and C', where B' is the path from v_1 to v_2 inclusive. If there is no C_*^{+2} subgraph, then there can be at most one edge from B' to P, and by Lemma 8 there can be at most 4 edges from either A' or C' to P (see Figure 3(a)). As there are 8 edges in total, there must be at least one edge to both A' and C'. Working off P' we also have edges e_3 and e_4 where $v_3 \in A'$ and $v_4 \in C'$ are furthest apart on P'. Ordering the vertices in each path left to right we may now assume $v_3 < v_1, v_2 < v_4$ and $u_1 < u_3, u_4 < u_2$. We may also assume without loss of generality that u_1, u_2 and v_3, v_4 are the end-vertices of P and P' respectively. If $u_4 < u_3$ and $v_1 < v_2$ then we obtain a cycle through all vertices except those between u_3 and u_4 and those between v_1 and v_2 (see Figure 3(b)). However each of these intervals meets at most one edge e_i, $i > 4$, so this cycle has at least $8 - 4 - 2 = 2$ chords. Thus, by reversing one path if necessary and relabeling the edges, we may assume that $u_3 \leq u_4$ and $v_1 \leq v_2$. As with P', decompose P into three paths A, B, C, where B consists of the vertices from u_3 to u_4 inclusive.

By considering the cycles $ABC'B'$ and $BCB'A'$ we see that there is at most one edge from AB to $B'C'$ and at most one edge from BC to $A'B'$. Thus there are two edges in total that either go between A and A' or go between C and C'. However by considering the cycles $Au_3A'v_1$ and $Cu_4C'v_2$ we see that both edges cannot go between the same pair, so there is an edge e_5 from A to A' and an edge e_6 from C to C'. On P we have $u_1 \leq u_5 < u_3 \leq u_4 < u_6 \leq u_2$ and on P' we have $v_3 \leq v_5 < v_1 \leq v_2 < v_6 \leq v_4$ (see Figure 3(c)).

There are two remaining edges, e_7 from AB to $B'C'$ and e_8 from BC to $A'B'$. Suppose that either $u_7 \in B$ or $v_8 \in B'$ or u_7 and v_8 lie on the same side of e_5. Then there is a path from B to B' through $A \cup A'$ meeting both u_7 and v_8 (see Figure 3(d)). If a similar situation holds for u_8 and v_7 on the other side of the graph then we can combine these paths to obtain a cycle through B and B' with e_7 and e_8 as chords. Thus without loss of generality we may assume that $u_1 \leq u_7 < u_5$ and $v_5 < v_8 < v_1$.

We may assume $v_7 > v_1$ and $u_8 > u_3$ as otherwise there would be a cycle through e_1 and e_3 with two chords. If $v_7 \geq v_6$ and $u_8 \leq u_6$ then there is a cycle with chords e_5 and e_6 as shown in Figure 3(e). If $v_7 \geq v_6$ and $u_8 \geq u_6$ then there is a cycle with chords e_5 and e_8 as shown in Figure 3(f). Finally, if $v_7 < v_6$ then there is a cycle with chords e_5 and e_7 as shown in Figure 3(g).

If the C_*^{+2} obtained uses all the vertices of both paths, then it forms a cycle with $(|P|-1)+(|P'|-1)+8-(|P|+|P'|) = 6$ chords. Thus by Lemma 5 there is a smaller $C_*^\dagger$ as a subgraph. Hence $P_* \cup P_* + 8e \supset C_*^\dagger$.

To prove the results of the form $P_n \cup P_m + ke \supset C_*^\dagger$ for $n < 8$ and all m, it is enough to consider $m \leq 2k-1$ with the edges between P_n and P_m meeting the first and last vertices of P_m and at least one of any pair of adjacent vertices on $P_m = v_1 \dots v_m$. This is because any counterexample with the k added edges missing vertices v_i and v_{i+1}, or missing v_1 or v_m, can be converted by the removal of a vertex of P_m into a counterexample with the path P_m replaced with P_{m-1}. Hence the remaining cases reduce to a finite case analysis. These cases were checked by computer yielding the stated results (see [9]). □

Lemma 10. *For $n \geq 5$ and $m \leq n$, $C_n \cup C_m + (2n+1)e \supset C_* \cup C_*^\dagger$. Also* $C_4 \cup C_4 + 11e \supset C_3 \cup C_4^+$, $C_4 \cup C_4 + 9e \supset C_* \cup C_*$, $C_4 \cup C_3 + 9e \supset C_3 \cup C_3$.

Proof. Drop the condition that $m \leq n$ and assume we have $2\max(n,m)+1$ edges between C_n and C_m and $\max(n,m) \geq 5$. For $\max(n,m) < 8$, the

Table 2: Minimum k such that $C_n \cup C_m + ke \supset C_* \cup C_*^\dagger$.

$n \backslash m$	3	4	5	6	7
4	–	11			
5	10	10	11		
6	11	13	11	11	
7	10	11	11	12	13

Also, $C_4 \cup C_4 + 9e \supset C_* \cup C_*$, $\quad C_4 \cup C_3 + 9e \supset C_3 \cup C_3$.

minimum number of edges needed to give a C_* and a $C_*^\dagger$ with fewer total vertices was found by exhaustive computer search (see [9]). The results are given in Table 2. Hence from now on we shall assume $\max(n, m) \geq 8$.

Let v be a vertex meeting the maximal number of edges between the two cycles and assume without loss of generality that $v \in C_n$ with v adjacent to d vertices in C_m. As the total number of edges between the cycles is more than $2n$, we have $3 \leq d \leq m$.

Case 1: $d \geq 6$.
Divide C_m into three vertex disjoint arcs P_1, P_2, and P_3, so that each P_i contains at least two neighbors of v. If there are at least 3 edges from P_i to $P = C_n - v$ then one obtains a C_* on $P_i \cup P$ not using all these vertices. (Two edges between two paths are enough to form a cycle, and if this cycle is hamiltonian then a third edge forms a chord, and hence one can find a shorter cycle.) However, there are at least 4 edges from v to $C_m - P_i$ forming a C_*^{+2} disjoint from this $P_i \cup P$. Hence we may assume each P_i sends at most 2 edges to $C_n - v$. The total number of edges between the cycles is then at most $3 \times 2 + d \leq m + 6$, which is less than $2\max(n, m) + 1$ when $\max(n, m) \geq 8$, a contradiction.

Case 2: $d = 5$.
Divide C_m into five vertex disjoint arcs $P_1, \ldots, P_5$, so that each P_i contains one neighbor of v. Since there are 4 edges to each $C_m - P_i$, we can as in Case 1 assume there are at most 2 edges from each P_i to $C_n - v$. This gives at most $5 \times 2 + d = 15$ edges between the cycles, which is less than $2\max(n, m) + 1$ when $\max(n, m) \geq 8$.

Case 3: $d = 4$.

Divide C_m into two vertex disjoint paths P_1, P_2, each containing two neighbors of v. Each P_i forms a cycle with v. If there are $|P_j| + 4$ edges from P_j to $P = C_n - v$, $j \neq i$, then by Lemma 9 we obtain a $C_*^\dagger$ on these vertices which does not use all the vertices of $P_j \cup P$ and is disjoint from a cycle in $P_i \cup \{v\}$. Thus we may assume there are at most $|P_j| + 3$ edges from each P_j to P. The total number of edges between the cycles is then at most $\sum_{i=1}^{2}(|P_i| + 3) + d = m + 10$. If $m > 6$ then we can in fact choose the paths P_i so that the end-vertices of each P_i are not both neighbors of v and so the cycle in $P_i \cup \{v\}$ does not use all the vertices of P_i. Then by Lemma 9 we can assume there are at most $|P_j| + 2$ edges from each P_j to P and we obtain a bound of $\sum_{i=1}^{2}(|P_i| + 2) + d = m + 8$ on the number of edges between the cycles. Thus in general there are at most $\max(m + 8, 16)$ edges between the cycles which is less than $2\max(n, m) + 1$ when $\max(n, m) \geq 8$.

Case 4: $d = 3$.
Let v_1, v_2, v_3 be the neighbors of v on C_m, and let P_i be the arc strictly between v_i and v_{i+1} (where $v_4 = v_1$). Let p_i be the number of edges from P_i to $P = C_n - v$ and n_i the number of edges from v_i to P. We can form a cycle $vv_1P_1v_2v$, so if there are more than 7 edges from $P_2v_3P_3$ to P then we are done by Lemma 9. Hence we may assume $p_2 + p_3 + n_3 \leq 7$. Adding the three cyclic rearrangements of this inequality gives $2\sum p_i + \sum n_i \leq 21$. But as the maximum number of edges meeting a vertex is $d = 3$, we have $n_i \leq 2$. (We are not counting the edge from v_i to v in n_i.) Hence $2\sum p_i + 2\sum n_i \leq 21 + 6 = 27$, so $\sum p_i + \sum n_i \leq 13$. Hence there are at most $13 + d = 16$ edges between the cycles, which is less than $2\max(n, m) + 1$ when $\max(n, m) \geq 8$. The lemma now follows. □

Lemma 11. *For $n \geq 6$ and $4 \leq m \leq n$, $C_n \cup C_m + (3n + 1)e \supset C_*^\dagger \cup C_*^\dagger$.*

Proof. Drop the condition that $m \leq n$ and assume we have $3\max(n, m) + 1$ edges between C_n and C_m and $\max(n, m) \geq 6$. For the cases when $\max(n, m) \leq 7$ the minimum number of edges needed to give two vertex disjoint $C_*^\dagger$ graphs with fewer total number of vertices was found by exhaustive computer search (see [9]). The results are given in Table 3. Hence from now on we shall assume $\max(n, m) \geq 8$.

Let v, v' be a pair of vertices that are adjacent on one of the cycles and which between them meet the maximum number of edges to the other cycle. Assume without loss of generality that $v, v' \in C_n$ with d_2 edges between

Table 3: Minimum k such that $C_n \cup C_m + ke \supset C_*^\dagger \cup C_*^\dagger$.

$n \backslash m$	4	5	6	7
5	14	16		
6	16	15	19	
7	15	17	17	18

$\{v, v'\}$ and C_m. As the total number of edges between the cycles is more than $3n$, we have $d_2 \geq 7$. A vertex $u \in C_m$ will be said to have *multiplicity* k, $k \in \{0, 1, 2\}$, if it is joined to k of the elements of $\{v, v'\}$. Let n_k be the number of multiplicity k vertices of C_m so that $m = n_0 + n_1 + n_2$ and $d_2 = n_1 + 2n_2$. Let P be an arc of C_m meeting some neighbors of $\{v, v'\}$. If the multiplicities of the neighbors along P contains one of the patterns

$$21, \qquad 2..2, \qquad 2..1..1, \qquad 2..1..2, \qquad 1..1..1..1..1, \tag{1}$$

(or their reflections) where .. denotes zero or more 0s, then there is a $C_*^\dagger$ in the graph using the vertices of $\{v, v'\} \cup P$. (The last two cases follow from Lemma 9 with $n = 2$, the others are easy exercises.)

Case 1: $d_2 > 10$.
Then either v or v' sends at least 6 edges to C_m, say v sends $d \geq 6$ such edges. Divide C_m into two (if $d \geq 8$) or three (if $d \in \{6, 7\}$) vertex disjoint arcs P_i so that each P_i meets at most $d - 4$ neighbors of v. If there are at least $|P_i| + 4$ edges from P_i to $P = C_n - v$ then by Lemma 9 one obtains a $C_*^\dagger$ on $P_i \cup P$ not using all these vertices. However, there are at least 4 edges from v to $C_m - P_i$ forming a $C_*^\dagger$ disjoint from this $C_*^\dagger$. Hence we may assume each P_i sends at most $|P_i| + 3$ edges to $C_n - v$. For $d \geq 8$ the total number of edges between the cycles is then at most $\sum_{i=1}^{2}(|P_i| + 3) + d = m + 6 + d \leq 2m + 6$ and for $d < 8$ the total number of edges between the cycles is at most $\sum_{i=1}^{3}(|P_i| + 3) + d = m + 9 + d \leq m + 16$. In both cases this is less that $3\max(n, m) + 1$ as $\max(n, m) \geq 8$, contradicting the assumption that there are $3\max(n, m) + 1$ edges between the cycles.

Case 2: $d_2 = 10$.
Assume first that one can divide C_m into two vertex disjoint arcs P_1 and P_2, so that each P_i meets at most $d_2 - 5 = 5$ edges from $\{v, v'\}$. If there are at least $|P_i| + 4$ edges from P_i to $P = C_n - \{v, v'\}$ then by Lemma 9 one obtains a $C_*^\dagger$ on $P_i \cup P$ not using all these vertices and a $C_*^\dagger$ on the disjoint subset

of vertices of the paths vv' and $C_m - P_i$. Thus the total number of edges between the cycles is at most $\sum_{i=1}^{2}(|P_i|+3)+d_2 = m+16$, which is less that $3\max(n,m)+1$ as $\max(n,m) \geq 8$. If it is not possible to find two such arcs, then the multiplicity pattern of points on C_m is 2..2..2..2..2... In this case we can still decompose C_m into two arcs P_i, each of whose complements includes the pattern 2..2 of (1). Thus we have a $C_*^\dagger$ on the vertices of $C_m - P_i$ and $\{v, v'\}$, and so the above argument still applies with this pair (P_1, P_2).

Case 3: $d_2 = 9$.
Assume first that $n_2 > 0$ and the multiplicity pattern of vertices along C_m is not 2..1..2..1..2..1.. with at least one multiplicity 0 vertex between consecutive neighbors of $\{v, v'\}$. Then we can find an arc P_1 of C_m with multiplicity pattern 21, 2..1..1, or 2..2 so that $P_2 = C_m - P_1$ sends at least 5 edges to $\{v, v'\}$. Thus, for $i = 1, 2$, there exists a $C_*^\dagger$ on the vertices of $C_m - P_i$ and $\{v, v'\}$. As in Case 2, we can now assume there are at most $|P_i|+3$ edges from P_i to $P = C_n - \{v, v'\}$ and so there are at most $\sum_{i=1}^{2}(|P_i|+3)+d_2 = m+15$ edges in total between the cycles. This gives a contradiction as $m + 15 < 3\max(n,m)+1$ when $\max(n,m) \geq 8$. Assume now that we are in one of the remaining cases where either we have the pattern 2..1..2..1..2..1.. with at least one multiplicity 0 vertex between consecutive neighbors of $\{v, v'\}$, or $n_2 = 0$. In both cases there are at least 9 vertices in C_m and we can decompose C_m into three vertex disjoint arcs P_i each sending 3 edges to $\{v, v'\}$. Then $C_m - P_i$ sends 6 edges to $\{v, v'\}$ and as above we can assume there are at most $\sum_{i=1}^{3}(|P_i|+3)+d_2 = m+18$ edges between the cycles. This gives a contradiction as $m+18 < 3\max(n,m)+1$ when $\max(n,m) \geq m \geq 9$.

Case 4: $d_2 = 8$.
If the multiplicity pattern on C_m is one of the following,

2..2..2..2.., 2..2..2..1..1.., 2..2..1..1..1..1.., 2..1..1..2..1..1..,

or if we have adjacent vertices with multiplicities 1 and 2, then we can decompose C_m into two arcs P_1, P_2, each containing one of the multiplicity patterns in (1). Thus each $C_m - P_i$ forms a $C_*^\dagger$ with $\{v, v'\}$. As above we get a bound of $\sum_{i=1}^{2}(|P_i|+3)+d_2 = m+14$ on the number of edges between the cycles. This gives a contradiction as $m+14 < 3\max(n,m)+1$ when $\max(n,m) \geq 8$. The remaining cases when $n_2 > 0$ are (up to cyclic rearrangements)

2..2..1..2..1.., 2..1..2..1..1..1.., 2..1..1..1..1..1..1..,

with multiplicity 0 vertices between the 1s and 2s. In each of these cases there are at least 9 vertices on C_m so $m \geq 9$. Also, in each case (and when

$n_2 = 0$) we can decompose C_m into three arcs P_i each meeting at most 3 edges to $\{v, v'\}$. Then $C_m - P_i$ sends at least 5 edges to $\{v, v'\}$, so as above we can assume there are at most $\sum_{i=1}^{3}(|P_i|+3)+d_2 = m+17$ edges between the cycles. This is less than $3\max(n,m)+1$ if $\max(n,m) \geq 9$, so we obtain a contradiction unless $n \leq m = 8$ and $n_2 = 0$. In this last case we may assume that each P_i contains either 2 or 3 vertices of C_m as there are no multiplicity 0 vertices on C_m. By Lemma 9 we can in fact assume each P_i sends at most 5 edges to $C_n - \{v, v'\}$ so our bound on the number of edges between the cycles is $3 \times 5 + d_2 = 23$ which is less than $3\max(n,m)+1 = 25$.

Case 5: $d_2 = 7$.
If $n_2 > 0$ then the multiplicity pattern on C_m is one of the following (up to cyclic rearrangements).

2..2..2..1.., 2..2..1..1..1.., 2..1..2..1..1.., 2..1..1..1..1..1..

In each case we can decompose C_m into three arcs P_i such that $C_m - P_i$ always contains one of the patterns of (1). As above the total number of edges between the cycles is at most $\sum(|P_i|+3)+d_2 = m+16$ which is less than $3\max(n,m)+1$ when $\max(n,m) \geq 8$. Hence we may assume $n_2 = 0$.

Let $v_1, \ldots, v_7$ be the neighbors of $\{v, v'\}$ on C_m arranged in a cyclic order, and let P_i be the arc of C_m from v_i through v_{i+1} to just before v_{i+2} (indices taken mod 7). Thus $P_1, \ldots, P_7$ form a double cover of C_m. Each $C_m - P_i$ sends 5 edges to $\{v, v'\}$ so as above we may assume each P_i sends at most $|P_i|+3$ edges to $C_n - \{v, v'\}$. Thus the total number of edges between the cycles is at most $\frac{1}{2}\sum_{i=1}^{7}(|P_i|+3)+d_2 = m+17\frac{1}{2}$. If $\max(n,m) \geq 9$ we are done as this is less than $3\max(n,m)+1$. Hence we may now assume $\max(n,m) = 8$. In this case there is at most one multiplicity 0 vertex on C_m so all P_i have either 2 or 3 vertices. Then by Lemma 9 we may assume that there are at most 5 edges from P_i to $C_n - \{v, v'\}$ and we obtain a bound of $\frac{1}{2}\sum_{i=1}^{7} 5 + d_2 = 24\frac{1}{2}$ on the total number of edges between the cycles. This is a contradiction as $3\max(n,m)+1 = 25$ in this case.

The lemma now follows as $d_2 \geq 7$. □

Theorem 12. *Suppose G is a graph with minimum degree $\delta(G) \geq 2$. Suppose further that if there is more than one vertex of degree 2 in G then the degree 2 vertices of G induce a path in G. Then G contains a cycle with at least two non-incident chords.*

We never use the fact that the chords found in Theorem 12 are non-incident, however we include this in the statement as it is a natural consequence of the proof.

Proof. By considering any single component of G, we may assume without loss of generality that G is connected. Let P_0 be the subgraph of G induced by the degree 2 vertices, so that P_0 is either a path, a single vertex, or empty. Suppose G is not 2-connected and let $B_1, \ldots, B_r$ be the blocks in the block cut-vertex decomposition of G. Further, suppose that if $P_0 \neq \emptyset$ then it meets B_1. Take any leaf-block $B_i \neq B_1$. Then B_i is not a single edge (as G has no degree 1 vertices), and can meet P_0 in at most one vertex (the cutvertex joining B_i to the rest of the graph). Hence B_i is 2-connected and all vertices in B_i have degree at least 3 except for the cut vertex joining B_i to the rest of G, which has degree at least 2 in B_i. By replacing G with B_i we may therefore assume G is 2-connected.

Since G is non-trivial and 2-connected, it contains a cycle. Pick a longest cycle C in G. The graph $G \setminus C$ is a union of components $S_1, \ldots, S_r$. For each chord uv of C we also include a fictitious empty component S_i that we declare to be joined to u and v only. In this way, each vertex $v \in C$, $v \notin P_0$, must be joined to at least one S_i as $d(v) \geq 3$. If two neighboring vertices u, v on C are joined to the same S_i then we can construct a longer cycle by replacing the edge uv of C by a path through S_i (which in this case is necessarily non-empty). However, each S_i must be joined to at least two vertices of C as G is 2-connected. We shall construct a new cycle with two non-incident chords using paths P_i joining vertices of C through the S_i, and edges of C. The chords will themselves be original edges of C.

For the rest of the proof we shall drop the assumption on the internal structure of the S_i, and use only the fact that two vertices on C joined to the same S_i can be joined by a path through S_i. Also, all vertices on C are joined to some S_i, except possibly those on a proper arc P_0 of C, and each S_i is joined to at least two vertices of C, no pair of which are adjacent on C. This slight generalization will be used in the proof of Theorem 13 below. Fix an orientation of C and for $u, v \in C$ write $[u, v]$ for the arc from u to v clockwise around C including the endpoints u and v. We shall also write $(u, v]$, $[u, v)$, or (u, v) for arcs that do not include endpoints u, v, or both u and v respectively. For $x \in C$ we shall abuse notation slightly by writing S_x for some S_i that is joined to x, even though the choice of S_i may not be

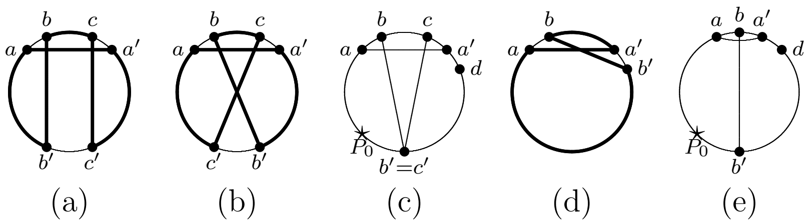

Figure 4: Proof of Theorem 12.

unique.

Pick two vertices a and a' that are joined to a common S_a and with minimal distance between a and a' in the path $C - P_0$. The assumptions above guarantee such a pair exists and that they are not adjacent on C. Let $[a, a'] = ab \dots ca'$ be the arc from a to a' in $C - P_0$. Assume first that $b \neq c$. By minimality of $[a, a']$, we may assume b and c are joined to distinct sets S_b and S_c, neither of which is S_a as S_a is not joined to neighboring vertices on C. Let $b' \neq b$ and $c' \neq c$ be vertices of C joined to S_b and S_c respectively, and for $x \in \{a, b, c\}$ let P_x be a path through S_x joining x to its corresponding x'. By minimality of $[a, a']$, b' and c' do not lie in the arc $[a, a']$. If $b' \neq c'$ then there is a cycle with chords ab and ca' (see Figure 4(a) and (b)). If $b' = c'$ and this vertex is adjacent to a or a', say a', then by Figure 4(d) we obtain a cycle with two chords ab and $a'b'$. Thus we may assume $b' = c'$ and this vertex is not a neighbor of a or a' on C. Also, we can assume that neither S_b nor S_c is joined to any vertex not in $\{b, c, b'\}$, so in particular they are not joined to any vertex in $[a', b')$. If $b = c$ then we have the situation in Figure 4(e), where once again we may assume any other vertex $b' = c'$ joined to S_b is not a neighbor of a or a' and, by suitable choice of $b' = c'$, we can again assume $S_b = S_c$ is not joined to any vertex in $[a', b')$. In the following we shall consider the case when $b \neq c$ and will only mention the $b = c$ case when there are differences in the proof.

As $[a, a']$ is disjoint from P_0, we may assume without loss of generality that P_0 (if non-empty) lies in (b', a) and hence that the vertex $d \in [a', b')$ adjacent to a' is not of degree 2 and thus is joined to some S_d. Now $S_d \neq S_a$ as otherwise S_a would be joined to neighboring vertices a', d on C, and $S_d \neq S_b, S_c$ as both S_b and S_c are not joined to any vertex in $[a', b')$, which includes the vertex d. Let $d' \neq d$ be a vertex on C joined to S_d. Then d' does not lie in the arc $(b, a']$ by minimality of $[a, a']$. Also $d' \neq b$ as otherwise we could use

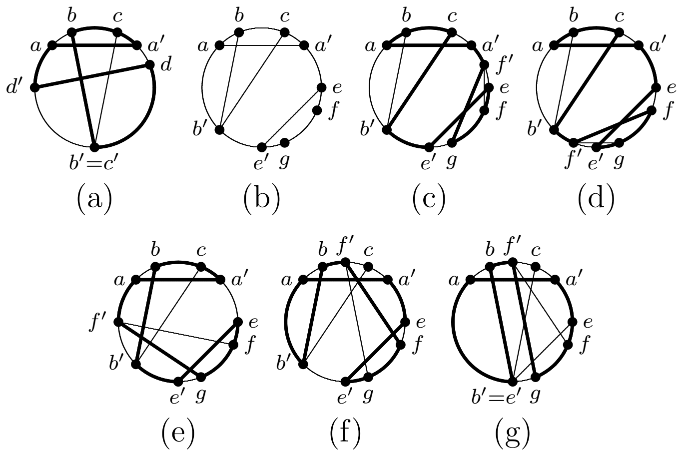

Figure 5: Proof of Theorem 12.

$\{b, d\}$ in place of $\{b, b'\}$ to obtain Figure 4(d). If $d' \in (b', a]$ then we have a cycle with chords ab and $a'd$ (Figure 5(a)). Hence we may assume $d' \in [d, b']$. Now choose $e, e' \in [d, b']$ joined to the some $S_e \notin \{S_a, S_b, S_c\}$ with minimal arc-length $[e, e']$. As d, d' have the required properties that $d, d' \in [d, b']$ and $S_d \notin \{S_a, S_b, S_c\}$, such a pair e, e' must exist. From now on we shall ignore $\{d, d'\}$ and work instead with $\{e, e'\}$, although these may be the same pair. Let $[e, e'] = ef \ldots ge'$, possibly with $f = g$. Note $[e, e']$ is disjoint from P_0, so f and g are joined to some S_f and S_g respectively. We have $S_f, S_g \notin \{S_b, S_c\}$ as S_b and S_c are not joined to any vertex in $[a', b')$, which includes both f and g. Also $S_f, S_g \neq S_e$ as otherwise S_e would be joined to adjacent vertices of C. Now both f and g are joined via S_f and S_g to vertices f' and g' respectively outside of $[e, e']$ for otherwise $f' \in [e, e']$ or $g' \in [e, e']$ would contradict the minimality of $[e, e']$. (If $S_f = S_a$, say, then this is automatic as we can take $f' = a$.) If precisely one of S_f and S_g are equal to S_a, say $S_f = S_a$, then we can take $f' = a$ or $f' = a'$ so that $f' \neq g'$. Thus we are in the case of Figure 4(a) or (b) with $(\{e, e'\}, \{f, f'\}, \{g, g'\})$ in place of $(\{a, a'\}, \{b, b'\}, \{c, c'\})$. If $S_f = S_g = S_a$ and $e' \neq b'$ then we are in the case of Figure 4(a) with $(\{a, f\}, \{b, b'\}, \{e, e'\})$ in place of $(\{a, a'\}, \{b, b'\}, \{c, c'\})$. If $S_f = S_g = S_a$ and $e' = b'$ then we are in the case of Figure 4(d) with $\{a, g\}$ taking the place of $\{a, a'\}$. Hence we may assume $S_f, S_g \notin \{S_a, S_b, S_c, S_e\}$.

If $f \neq g$ then $S_f \neq S_g$ (by minimality of $[e, e']$) and so arguing as above with $\{e, e'\}$ in place of $\{a, a'\}$ we can assume $f' = g'$. If $f = g$ then we also choose $f' = g'$. If $f' \in [a', e)$ then we have a cycle with chords ca' and $e'g$

(Figure 5(c)). If $f' \in (e', b']$ then we have a cycle with chords ca' and ef (Figure 5(d), note $a' \neq e$ so chords do not intersect). If $f' \in (b', a]$ then we have a cycle with chords ab and $e'g$ (Figure 5(e)). If $f' \in (a, a')$ and $e' \neq b'$ then we have a cycle with chords ab and ef (Figure 5(f)). Finally, if $f' \in (a, a')$ and $e' = b'$ then we have a cycle with chords ab and $e'g$ (Figure 5(g)). □

Theorem 13. *Suppose G contains a cycle C and every vertex in $S = G - C$ has degree at least 3 in G and $S \neq \emptyset$. Then either G contains a cycle shorter than C, or G contains a cycle with two chords.*

Proof. By restricting to a single component of $G[S]$ we may assume $G[S]$ is connected. If S consists of a single vertex then it sends at least three edges to C, so we are done by Lemma 3. Hence we may assume S contains at least two vertices. Consider the block cut-vertex decomposition of $G[S]$. Suppose there is a leaf-block B, possibly joined to the rest of $G[S]$ via a cut-vertex v_1 and suppose that there are no edges from $B - v_1$ to C in G. Then every vertex in B except possibly v_1 is of degree at least 3 in $G[B]$, so by Theorem 12 we have a C_*^{+2} in $G[B]$. Hence we may assume that there is an edge from $B - v_1$ to C.

Each leaf block is either a single edge or is 2-connected. Suppose first that there exists a 2-connected leaf-block B. If $B \neq S$ let v_1 be the cut-vertex joining B to the rest of S in $G[S]$. Since every other leaf-block is joined to C, we may assume there is a path P from v_1 to C that does not meet $B - v_1$. If $B = S$ then we can set v_1 to be any vertex of B joined to C and P to be the single edge path joining v_1 to C. Then in the graph $G[B \cup P \cup C]$, each vertex of B has degree at least 3.

Pick a maximal cycle C' in $G[B]$. Then as in the proof of Theorem 12, we can decompose $G[B] - C'$ into components S_i. If $(P - v_1) \cup C$ is adjacent to vertices in C' we shall consider this an extra component, which will be denoted S_1. Chords of C' will be associated to fictitious empty components S_i. Now each vertex of C' is joined to some S_i, and each $S_i \neq S_1$ is joined to at least two vertices of C'. Moreover, each $S_i \neq S_1$ cannot be joined to adjacent vertices on C' by maximality of C'. The component S_1 however may be joined to any number of vertices of C' and even to adjacent vertices of C' as the cycle C' was chosen to be maximal in $G[B]$ rather than in $G[B \cup P \cup C]$. By removing any edges to S_1 from each $v \in C'$ that is joined to some $S_i \neq S_1$, we may

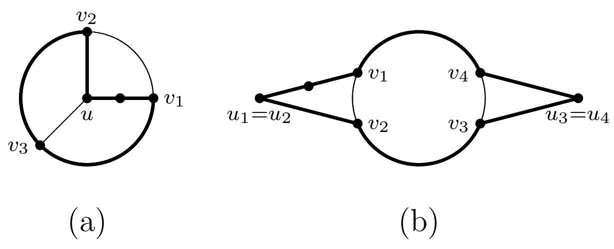

Figure 6: Proof of Theorem 13.

assume that any vertex $v \in C'$ that is joined to S_1 is joined to no other S_i. If S_1 is now joined only to the vertices of C' forming an proper arc of C', then we denote this arc as P_0, remove all connections to S_1 and proceed with the proof as in Theorem 12. (This includes the case when S_1 is joined to zero or one vertex of C', but not the case when S_1 is joined to all vertices of C, which will be dealt with below.) If the set of vertices joined to S_1 forms just one arc plus some other isolated vertices on C', then we let P_0 contain all but one end-vertex of the arc. Remove connections from P_0 to S_1. Then S_1 is now joined to at least two vertices of C', but is not joined to any pair of adjacent vertices on C'. In this case we can proceed as in Theorem 12. Now suppose there are two non-trivial arcs of vertices joined to S_1. Then in particular there are at least 4 vertices of C' joined to S_1. Thus we may assume that C and C' are joined by 3 edges and a path P (which may itself be a single edge), and these meet two pairs of adjacent vertices on C'. The only other case is when S_1 is joined to all vertices of C'. In both these cases, let v_1Pu_1, v_2u_2,..., v_ku_k, $k \in \{3, 4\}$ be paths and edges from C' to C with v_1 adjacent to v_2, and v_3 either adjacent to v_4 (if $k = 4$) or adjacent to both v_1 and v_2 (if $k = 3$). The last case is needed only if C' is a triangle and S_1 is joined to all vertices of C'. If $u_1 \neq u_2$ then one of the arcs from u_1 to u_2 on C contains u_3 and we obtain a cycle $v_1 \dots v_3 \dots v_2u_2 \dots u_3 \dots u_1Pv_1$ with chords v_1v_2 and v_3u_3. Thus we may assume $u_1 = u_2$. Similarly if $k = 4$ we may assume $u_3 = u_4$ (as otherwise we would obtain a cycle with chords v_3v_4 and v_2u_2). If $k = 3$ we may assume (by interchanging v_2, u_2 and v_3, u_3) that $u_1 = u_2 = u_3$. If $u_1 = u_2 = u_3 = u$ then we have a cycle $v_1Pu_1v_2 \dots v_3 \dots v_1$ with chords v_1v_2 and u_3v_3 (see Figure 6(a)). If $u_1 = u_2 \neq u_3 = u_4$ then we have a cycle $u_1Pv_1 \dots v_4u_4v_3 \dots v_2u_1$ with chords v_1v_2 and v_3v_4 (Figure 6(b)).

Now suppose there are no 2-connected leaf blocks of $G[S]$. Then either $G[S]$ is a single edge, or its block cut-vertex decomposition contains at least two leaf blocks which are single edges. In either case there is a path P joining two vertices v_1 and v_2 of degree 1 in $G[S]$. Since v_1 and v_2 have degree at least 3 in G, each must be joined to two vertices of C. Denote the neighbors of $\{v_1, v_2\}$ in C as $u_1, \ldots, u_k$, where $2 \leq k \leq 4$ and the u_i are arranged cyclically around C. There must be some consecutive pair, say u_1, u_2 such that u_1 is joined to v_1 and u_2 is joined to v_2. Then $v_1 P v_2 u_2 \ldots u_3 \ldots u_k \ldots u_1 v_1$ is a cycle with two chords $v_1 u_i$ and $v_2 u_j$ for some $i, j \in \{1, \ldots, k\}$. □

References

[1] A. Bialostocki, D. Finkel, and A. Gyárfás, Disjoint chorded cycles in graphs, *Discrete Math.* **308** (2008), 5886–5890.

[2] S. Chiba, S. Fujita, Y. Gao, and G. Li, On a sharp degree sum condition for disjoint chorded cycles in graphs, *Graphs and Combinatorics* **26** (2010), 173–186.

[3] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423–439.

[4] Y. Gao, G. Li, J. Yan, Disjoint cycles and chorded cycles in graphs, *Proc. Intr. Conf. on Foundation of Computer Science*, Las Vegas, Nevada, 2008, 9–13.

[5] Y. Gao, G. Li, J. Yan, Neighborhood unions for the existence of disjoint chorded cycles in graphs. *Graphs Combin.* **29** (2013), 1337–1345.

[6] R. Gould, K. Hirohata, and P. Horn, On independent doubly chorded cycles. *Discrete Math.* **338** (2015), 2051–2071.

[7] L. Lovász, Combinatorial Problems and Exercises, North Holland, 1993.

[8] S. Qiao, Neighborhood unions and disjoint chorded cycles in graphs. Discrete Math. **312** (2012), 891–897.

[9] Supplemental material including computer code and results can be found at `http://memphis.edu/msci/people/pbalistr/cyclechord.php`.