PACKING DIGRAPHS WITH DIRECTED CLOSED TRAILS

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Abstract. It has been shown [Balister, 2001] that if n is odd and $m_1, \dots, m_t$ are integers with $m_i \geq 3$ and $\sum_{i=1}^{t} m_i = |E(K_n)|$ then K_n can be decomposed as an edge-disjoint union of closed trails of lengths $m_1, \dots, m_t$. This result was later generalized [Balister, to appear] to all sufficiently dense Eulerian graphs G in place of K_n. In this article we consider the corresponding questions for directed graphs. We show that the compete directed graph $\overleftrightarrow{K}_n$ can be decomposed as an edge-disjoint union of directed closed trails of lengths $m_1, \dots, m_t$ whenever $m_i \geq 2$ and $\sum m_i = |E(\overleftrightarrow{K}_n)|$, except for the single case when $n = 6$ and all $m_i = 3$. We also show that sufficiently dense Eulerian digraphs can be decomposed in a similar manner, and we prove corresponding results for (undirected) complete multigraphs.

1. Introduction

All graphs considered in the first three sections will be finite simple graphs or digraphs (without loops or multiple edges). Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph or digraph G. If G is a graph, $\overleftrightarrow{G}$ will denote the digraph obtained from G be replacing each edge $xy \in E(G)$ by the pair of directed edges $\vec{xy}$ and $\vec{yx}$. We shall often identify G and $\overleftrightarrow{G}$ when there is no danger of confusion. We say a graph (digraph) G is *Eulerian* iff it has a (directed) closed trail through every edge of G. Equivalently, G is connected and either has even degree $d_G(v)$ at each vertex (for simple graphs) or the in-degree and out-degree of each vertex v are the same $d_G^-(v) = d_G^+(v)$ (for digraphs).

Write $n = |V(G)|$ for the number of vertices of G. If $S \subseteq E(G)$ write $G \setminus S$ for the graph with the same vertex set as G, but edge set $E(G) \setminus S$. Sometimes we shall abuse notation by writing, for example, $G \setminus H$ for $G \setminus E(H)$ when H is a subgraph of G.

In Section 2 we shall prove the first main result:

Theorem 1.1. *If $\sum_{i=1}^{t} m_i = n(n-1)$ and $m_i \geq 2$ for $i = 1, \dots, t$ then $\overleftrightarrow{K}_n$ can be decomposed as the edge-disjoint union of directed closed trails of lengths $m_1, \dots, m_t$, except in the case when $n = 6$ and all $m_i = 3$.*

In [2] the analogous theorem was proved for the simple graphs K_n, n odd, and $K_n - I$, n even, where I is a 1-factor of K_n and all $m_i \geq 3$. Packing directed *cycles* into $\overleftrightarrow{K}_n$ has also been studied by Alspach et al., [1] when all the m_i are equal. In this case we clearly need $2 \leq m_i = m \leq n$ and $m \mid n(n-1)$. Packings exist for all such pairs (m, n) except $(m, n) = (4, 4)$, $(3, 6)$, or $(6, 6)$.

The strategy used in the proof of Theorem 1.1 is to first pack closed trails of arbitrary lengths into graphs formed by linking small complete digraphs together.

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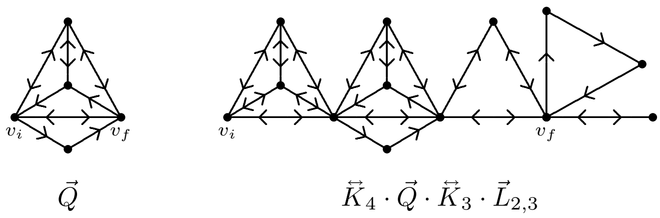


FIGURE 1. The graph $\vec{Q}$ and an example of linked digraphs.

We then decompose $\overleftrightarrow{K}_n$ for large n into linked complete digraphs. Finally we need to treat some small values of n specially.

In Section 3 we generalize Theorem 1.1 to:

Theorem 1.2. *There exist absolute constants N and $\epsilon > 0$ such that for any Eulerian digraph G with $|V(G)| \geq N$, $\delta^+(G) = \delta^-(G) \geq (1-\epsilon)|V(G)|$, and for any $m_1, \dots, m_t$ with $m_i \geq 3$, $\sum_{i=1}^t m_i = |E(G)|$, G can be written as the edge disjoint union of directed closed trails of lengths $m_1, \dots, m_t$. Moreover, if $G = \overleftrightarrow{H}$ for some simple graph H, then $m_i \geq 2$ is sufficient.*

Finally in Section 4 we conclude by giving necessary and sufficient conditions for packing complete multigraphs with closed trails.

2. PACKING $\overleftrightarrow{K}_n$ WITH DIRECTED CLOSED TRAILS

If G_1 and G_2 are graphs or digraphs, define a *packing* of G_1 into G_2 as a map $f\colon V(G_1) \to V(G_2)$ such that $\vec{xy} \in E(G_1)$ implies $\overrightarrow{f(x)f(y)} \in E(G_2)$ and the induced map on edges $\vec{xy} \mapsto \overrightarrow{f(x)f(y)}$ is a bijection between $E(G_1)$ and $E(G_2)$. Note that f is *not* required to be injective on vertices. Hence if G_1 contains a path or cycle, its image in G_2 will be a trail or closed trail. With this notation, the problem is one of packing a disjoint union of directed cycles into $\overleftrightarrow{K}_n$.

We shall define for certain graphs or digraphs initial and final *link* vertices. If G_1 and G_2 are graphs or digraphs for which such vertices are specified, define $G_1 \cdot G_2$ as the graph obtained by identifying the final link vertex of G_1 with the initial link vertex of G_2. The initial link vertex of $G_1 \cdot G_2$ will be the same as the initial link for G_1 and the final link vertex will be the same as the final link for G_2. This makes '$\cdot$' into an associative operation on graphs when defined. Similarly we define the initial link vertex of the vertex-disjoint union $G_1 \cup G_2$ to be that of G_1 and the final link vertex to be that of G_2.

For K_n or $\overleftrightarrow{K}_n$, $n \geq 2$, define the initial and final link vertices as any two distinct vertices. Let $\vec{C}_n$ be a directed cycle on n vertices. Let the digraph $\vec{L}_{a_1,\dots,a_r}$ consist of directed cycles of lengths $a_1, \dots, a_r$ intersecting in a single vertex v, which will be both the initial and final link for this graph. In the special case when $r = 0$ write $\vec{L}_0$ for a single isolated vertex v. More generally, we ignore any a_i that are zero. Note that there exists a packing $\vec{L}_{a+b} \to \vec{L}_{a,b}$ for any $a, b \geq 2$ preserving the link vertex. Define the directed graph $\vec{Q}$ as shown in Figure 1. The initial and final links are indicated by v_i and v_f respectively. The graph $\vec{Q}$ is obtained from $\overleftrightarrow{K}_4$ by 'splitting' off a directed path of length two joining the link vertices.

Definition 1. *Let $\mathcal{S}$ be the set of digraphs consisting of $\vec{L}_0$, $\vec{L}_2$, $\vec{L}_3$, $\vec{L}_5$, $\vec{L}_{2,2}$, $\vec{L}_{3,3}$, $\vec{L}_{2,n}$ for $n \geq 4$, and $\vec{L}_{3,n}$ for $n \geq 5$.*

Note that for any $n \geq 2$ we can pack $\vec{L}_n$ into some graph $\vec{L} \in \mathcal{S}$, preserving the link vertex.

Theorem 2.1. *Suppose $\vec{L} \in \mathcal{S}$, $t \geq 0$, and $m_i \geq 2$, $m_i \neq 3$, for $i = 1, \dots, t$. Assume also that $\ell = |E(\vec{L})| + \sum_{i=1}^{t} m_i$ is even.*

(a) *If $\ell \geq 6$ then there exists a subset $S \subseteq \{1, \dots, t\}$ and a packing of $\vec{L}$ and directed cycles $\vec{C}_{m_i}$, $i \in S$, into some digraph of the form $\overleftrightarrow{K}_3 \cdot \vec{L}'$ with $\vec{L}' \in S$.*

(b) *If $\ell \geq 12$ then there exists a subset $S \subseteq \{1, \dots, t\}$ and a packing of $\vec{L}$ and directed cycles $\vec{C}_{m_i}$, $i \in S$, into some digraph of the form $\overleftrightarrow{K}_4 \cdot \vec{L}'$ with $\vec{L}' \in S$.*

(c) *If $\ell \geq 12$ and $m_i > 2$ for all i then there exists a subset $S \subseteq \{1, \dots, t\}$ and a packing of $\vec{L}$ and directed cycles $\vec{C}_{m_i}$, $i \in S$, into some digraph of the form $\vec{Q} \cdot \vec{L}'$ with $\vec{L}' \in S$.*

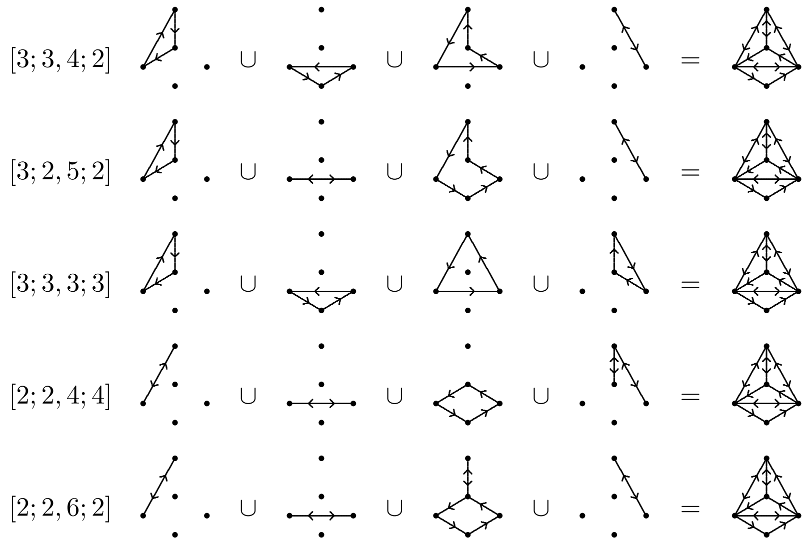
In all cases the packing maps the initial link of $\vec{L}$ to the initial link of the resulting graph.

Proof. In all cases, if $\vec{L} = \vec{L}_0$, $\vec{L}_2$, or $\vec{L}_3$, then $t > 0$ and we can pack $\vec{L}$ and some $\vec{C}_{m_i}$ into $\vec{L}_{m_i}$, $\vec{L}_{2,m_i}$, or $\vec{L}_{3,m_i}$ respectively. In the $\vec{L}_2$ case note that $m_i \neq 3$ and in the $\vec{L}_3$ case ℓ is even so we can choose m_i to be odd. Hence in all cases we can pack some larger $\vec{L}' \in \mathcal{S}$. Thus we are reduced to the cases when $\vec{L} = \vec{L}_5$, $\vec{L}_{2,2}$, $\vec{L}_{2,n}$ ($n \geq 4$), $\vec{L}_{3,3}$, or $\vec{L}_{3,n}$ ($n \geq 5$), possibly with a smaller value of t. We shall also use the fact that ℓ is even to deduce that there is an odd m_i whenever $|E(\vec{L})|$ is odd.

(a) Since $\overleftrightarrow{K}_3$ is the union of two $\vec{C}_3$s, we can pack $\vec{L}_{3,n}$ into $\overleftrightarrow{K}_3 \cdot \vec{L}_{n-3}$ for $n \geq 5$ or $n = 3$. The graph $\vec{L}_{n-3}$ can then be packed into some $\vec{L}' \in \mathcal{S}$. Similarly we can pack $\vec{L}_5$ and $\vec{C}_{m_i}$ (m_i odd) into $\overleftrightarrow{K}_3 \cdot \vec{L}_{2,m_i-3}$. The $\vec{L}_5$ is packed as a $\vec{C}_3$ inside $\overleftrightarrow{K}_3$ joined to to the $\vec{L}_2$ and $\vec{C}_{m_i}$ is packed as the other $\vec{C}_3$ in $\overleftrightarrow{K}_3$ joined to the $\vec{L}_{m_i-3}$. Regarding $\overleftrightarrow{K}_3$ as the union of three $\vec{C}_2$s, we can pack $\vec{L}_{2,n}$ as $\overleftrightarrow{K}_3 \cdot \vec{L}_{n-4}$ for $n = 4$ or $n \geq 6$. For $n = 5$ we have an odd $m_i \geq 5$ and so we can pack $\vec{L}_{2,5} \cup \vec{C}_{m_i}$ into $\overleftrightarrow{K}_3 \cdot \vec{L}_{3,m_i-2}$. For $n = 2$, $\vec{L}_{2,2}$ has fewer than 6 edges, so there is at least one m_i and we can pack $\vec{L}_{2,2} \cup \vec{C}_{m_i}$ into $\overleftrightarrow{K}_3 \cdot \vec{L}_{m_i-2}$ since $m_i \neq 3$. Hence in all cases we obtain a packing into $\overleftrightarrow{K}_3 \cdot \vec{L}'$ with $\vec{L}' \in \mathcal{S}$.

(b) Since $\vec{Q}$ can be packed into $\overleftrightarrow{K}_4$ with both links fixed, we are reduced to case (c) unless some of the $m_i = 2$. We now prove (c) without the restriction that $m_i > 2$ and check that in the cases when a packing is not possible with $m_i = 2$, the corresponding packing with $\overleftrightarrow{K}_4$ in place of $\vec{Q}$ exists.

(c) The graph $\overleftrightarrow{K}_3$ is a subgraph of $\vec{Q}$ (with the same link vertices). The edges of $\vec{Q} \setminus \overleftrightarrow{K}_3$ form a closed trail of length 6 meeting both link vertices. This closed trail is formed as a $\vec{C}_2$ and a $\vec{C}_4$ intersecting in a single vertex. As a result, if $m_i = 6$ for any i (or $m_i = 4$, $m_j = 2$), we can pack this cycle (or pair of cycles) into the closed trail $\vec{Q} \setminus \overleftrightarrow{K}_3$ and reduce to case (a). Similarly, some of the cases $\vec{L} = \vec{L}_{a,b}$ can be reduced to case (a) with $\vec{L} = \vec{L}_{a,b-6}$ by packing $\vec{L}_{a,b-6}$ and the $\vec{C}_{m_i}$ into $\overleftrightarrow{K}_3 \cdot \vec{L}'$ and then attaching the missing closed trail $\vec{Q} \setminus \overleftrightarrow{K}_3$ to the closed trail $\vec{L}_{b-6}$



$[2; 3, 4; 3]$	Reverse the trails in $[3; 3, 4; 2]$ and interchange link vertices.
$[2; 2, 5; 3]$	Reverse the trails in $[3; 2, 5; 2]$ and interchange link vertices.
$[; 2, 5, 5;]$	Combine the last $\vec{C}_2$ with the $\vec{C}_3$ in the $[3; 2, 5; 2]$ packing.

FIGURE 2. Packings used in Theorem 2.1

(both of which meet the initial vertex). Hence we are reduced to the cases when $\vec{L} \in \{\vec{L}_5, \vec{L}_{2,2}, \vec{L}_{2,4}, \vec{L}_{2,5}, \vec{L}_{2,7}, \vec{L}_{2,9}, \vec{L}_{3,3}, \vec{L}_{3,5}, \vec{L}_{3,7}, \vec{L}_{3,8}, \vec{L}_{3,10}\}$.

Define $[a_1, \dots, a_p; b_1, \dots, b_q; c_1, \dots, c_r]$ as a decomposition of $\vec{Q}$ into directed closed trails of lengths a_i, b_i, c_i, with the closed trails of lengths a_i meeting the initial link, the closed trails of lengths c_i meeting the final link, and the closed trails of lengths b_i meeting both link vertices. Figure 2 shows some such decompositions.

Table 1 describes most of the remaining packings. The various closed trails required are constructed by combining closed trails of $[\dots] \cdot \vec{L}'$ in each case. The underlined cycles in the first column are packed into the underlined closed trails in the second column. Recall that we are assuming $\ell \geq 12$, $m_i \neq 3, 6$, and if some $m_i = 4$ then no $m_j = 2$. Also any $\vec{L}_n$ can be packed into some $\vec{L} \in \mathcal{S}$. It is easy to see that we can use one of these packings except in the cases when $\vec{L} = \vec{L}_{3,3}$, $\vec{L}_{2,4}$, or $\vec{L}_{2,2}$, and all the m_i equal 2.

In these cases we must be in case (b) with at least three remaining m_i equal to 2. We can write $\vec{K}_4$ as a union of a $\vec{K}_3$ (meeting both link vertices) and three $\vec{C}_2$s (forming a 3-distar). Use the distar to pack the three $\vec{C}_2$s and remove these edges. We are then reduced to case (a). □

Let H be a simple graph with an edge-decomposition into triangles $\mathcal{T}$, so $E(H)$ is a disjoint union $\bigcup_{T \in \mathcal{T}} E(T)$ and each edge of H is in a unique triangle of $\mathcal{T}$. Define a *trail* of triangles as a sequence of triangles $T_1, \dots, T_n$ in $\mathcal{T}$ determined by a trail $P = e_1 \dots e_n$ (of edges) in H, where the edge e_i lies in T_i and the T_i are distinct triangles of $\mathcal{T}$. We call P the *underlying* trail. Define an *Eulerian trail*

TABLE 1. Packings used in Theorem 2.1

Case	Packed as	Conditions
$\vec{L}_{3,\underline{10}} \cup \vec{C}_{m_i}$	$[3; \underline{3}, \underline{4}; 2] \cdot \vec{L}_{\underline{3},m_i-2}$	m_i odd
$\vec{L}_{3,\underline{8}} \cup \vec{C}_{m_i}$	$[3; 3, \underline{4}; \underline{2}] \cdot \vec{L}_{\underline{2},m_i-3}$	m_i odd
$\vec{L}_{3,\underline{7}} \cup \vec{C}_{m_i}$	$[3; \underline{3}, \underline{4}; 2] \cdot \vec{L}_{m_i-2}$	all m_i
$\vec{L}_{3,5} \cup \vec{C}_{m_i}$	$[3; 2, 5; 2] \cdot \vec{L}_{m_i-4}$	$m_i \neq 2, 5$
$\vec{L}_{3,5} \cup \vec{C}_{\underline{5}} \cup \vec{C}_{m_i}$	$[3; \underline{2}, 5; 2] \cdot \vec{L}_{\underline{3},m_i-2}$	m_i odd
$\vec{L}_{3,5} \cup \vec{C}_{\underline{2}} \cup \vec{C}_{m_i}$	$[3; \underline{2}, 5; 2] \cdot \vec{L}_{m_i-2}$	all m_i
$\vec{L}_{3,3} \cup \vec{C}_{m_i}$	$[3; 3, 4; 2] \cdot \vec{L}_{m_i-6}$	$m_i \neq 2, 4, 5, 7$
$\vec{L}_{3,3} \cup \vec{C}_{\underline{7}} \cup \vec{C}_{m_i}$	$[3; 3, \underline{4}; 2] \cdot \vec{L}_{\underline{3},m_i-2}$	m_i odd
$\vec{L}_{3,3} \cup \vec{C}_{\underline{5}} \cup \vec{C}_{m_i}$	$[3; 3, \underline{3}; 3] \cdot \vec{L}_{\underline{2},m_i-3}$	m_i odd
$\vec{L}_{3,3} \cup \vec{C}_{\underline{4}} \cup \vec{C}_{m_i}$	$[3; 3, \underline{4}; 2] \cdot \vec{L}_{m_i-2}$	all m_i
$\vec{L}_{2,\underline{9}} \cup \vec{C}_{m_i}$	$[2; \underline{3}, \underline{4}; 3] \cdot \vec{L}_{\underline{2},m_i-3}$	m_i odd
$\vec{L}_{2,\underline{7}} \cup \vec{C}_{m_i}$	$[2; \underline{3}, \underline{4}; 3] \cdot \vec{L}_{m_i-3}$	m_i odd
$\vec{L}_{2,\underline{5}} \cup \vec{C}_{m_i}$	$[; 2, \underline{5}, 5;] \cdot \vec{L}_{m_i-5}$	m_i odd
$\vec{L}_{2,4} \cup \vec{C}_{m_i}$	$[2; 2, 4; 4] \cdot \vec{L}_{m_i-6}$	$m_i \neq 2, 4, 5, 7$
$\vec{L}_{2,4} \cup \vec{C}_{\underline{7}} \cup \vec{C}_{m_i}$	$[2; 2, 4; \underline{4}] \cdot \vec{L}_{\underline{3},m_i-2}$	m_i odd
$\vec{L}_{2,4} \cup \vec{C}_{\underline{5}} \cup \vec{C}_{m_i}$	$[2; \underline{3}, 4; 3] \cdot \vec{L}_{\underline{2},m_i-3}$	m_i odd
$\vec{L}_{2,4} \cup \vec{C}_{\underline{4}} \cup \vec{C}_{m_i}$	$[2; 2, 4; \underline{4}] \cdot \vec{L}_{m_i-2}$	all m_i
$\vec{L}_{2,2} \cup \vec{C}_{m_i}$	$[2; 2, 4; 4] \cdot \vec{L}_{m_i-8}$	$m_i \neq 2, 4, 5, 7, 9$
$\vec{L}_{2,2} \cup \vec{C}_{\underline{9}} \cup \vec{C}_{m_i}$	$[2; 2, \underline{6}; 2] \cdot \vec{L}_{\underline{3},m_i-2}$	m_i odd
$\vec{L}_{2,2} \cup \vec{C}_{\underline{7}} \cup \vec{C}_{m_i}$	$[2; 2, \underline{4}; 4] \cdot \vec{L}_{\underline{3},m_i-4}$	m_i odd, $\neq 5$
$\vec{L}_{2,2} \cup \vec{C}_{\underline{5}} \cup \vec{C}_{m_i}$	$[2; 2, \underline{5}; 3] \cdot \vec{L}_{m_i-3}$	m_i odd
$\vec{L}_{2,2} \cup \vec{C}_{\underline{4}} \cup \vec{C}_4$	$[2; 2, \underline{4}; 4] \cdot \vec{L}_0$	
$\vec{L}_5 \cup \vec{C}_{m_i}$	$[; 2, 5, 5;] \cdot \vec{L}_{m_i-7}$	m_i odd, $\neq 5$
$\vec{L}_5 \cup \vec{C}_5 \cup \vec{C}_{m_i}$	$[; 2, 5, 5;] \cdot \vec{L}_{m_i-2}$	all m_i

Note: in all cases $m_i \neq 3, 6$ and $\ell \geq 12$, ℓ even.

of triangles as a closed trail of triangles (i.e., P is a closed trail) which uses every triangle of $\mathcal{T}$. We say an Eulerian trail of triangles is *good* if the underlying closed trail P meets every vertex of H.

Recall a *Steiner Triple System* on K_n is an edge-decomposition of K_n into triangles. Steiner Triple Systems exist for all $n \equiv 1$ or 3 mod 6 [5]. A Steiner triple system is said to be *resolvable* if the triangles can be partitioned into classes with the triangles in each class forming a 2-factor of K_n. Resolvable Steiner Triple Systems exist for all $n \equiv 3$ mod 6 [6].

Lemma 2.2. *Let $\mathcal{T}$ be a Steiner Triple System on K_n. Then the triangles of $\mathcal{T}$ can be arranged into a trail of triangles. Moreover, if $n \geq 7$ then $\mathcal{T}$ can be arranged into a good Eulerian trail of triangles.*

Proof. Assume first that $n \geq 13$ and let $\mathcal{T} = \{T_1, \ldots, T_N\}$ be the Steiner Triple System on K_n. It is sufficient to construct an Eulerian subgraph G of K_n (with no isolated vertices) that contains precisely one edge from each triangle $T_i \in \mathcal{T}$. An Eulerian trail in G will then give a good Eulerian trail of triangles in K_n.

Pick one triangle, T_1 say, from $\mathcal{T}$. Let T_1 have vertex set $V(T_1) = \{r_1, r_2, r_3\}$ and let $M = V(K_n) \setminus V(T_1)$ be the remaining $n-3$ vertices of K_n. Let $\mathcal{T}_M$ be the subset of triangles $T_i \in \mathcal{T}$ that have all their vertices in M. Each vertex $v \in M$ meets precisely three triangles that are not in $\mathcal{T}_M$, one for each edge vr_j. Hence each $v \in M$ is incident to exactly $n-7$ edges that lie in triangles in $\mathcal{T}_M$. Let $S \subseteq \mathcal{T}_M$ and let m be the number of vertices in M meeting some triangle in S. The number of edges of triangles in S is $3|S|$, however this is at most $\frac{m}{2}(n-7)$ since each of these edges meets two of these m vertices. Thus $|S| \leq \lceil \frac{n-7}{6} \rceil m$. Assign to each triangle $T_i \in \mathcal{T}_M$ a vertex $v_i \in V(T_i)$ so that no more than $\lceil \frac{n-7}{6} \rceil$ triangles are assigned to each vertex of M. This is equivalent to finding a matching in a bipartite graph with one class equal to $\mathcal{T}_M$, and the other class consisting of $\lceil \frac{n-7}{6} \rceil$ copies of M, and edges joining T_i to all copies of the vertices that lie in $V(T_i)$. By Hall's theorem we can construct such a matching from triangles to vertices provided every set S of triangles meets at least $|S|$ copies of vertices. However S meets $\lceil \frac{n-7}{6} \rceil m \geq |S|$ such copies, so such a matching does indeed exist.

Construct a subgraph G of K_n consisting of one edge in M from each triangle T_i, $i \neq 1$. For each triangle $T_i \in \mathcal{T}_M$ we let G contain the unique edge of T_i that does not meet v_i. So far each vertex $v \in M$ has degree in G of at least $\frac{n-7}{2} - \lceil \frac{n-7}{6} \rceil \geq 2$ since there are $\frac{n-7}{2}$ triangles of $\mathcal{T}_M$ that meet v and each triangle that meets v other than those with $v_i = v$ contribute one to this degree. Hence the smallest component of G has size at least 3. We now show that we can choose edges from the triangles meeting r_3 so that $G[M \cup \{r_3\}]$ becomes connected.

Add edges from triangles meeting r_3 as follows. Pick any component of G that is not already connected to r_3. Pick some vertex v of this component. Add the edge vr_3 to G and let u be the other vertex of the triangle in $\mathcal{T}$ containing the edge vr_3. If u is in a new component of G not already joined to r_3, pick some $u' \neq u$ in this component and continue by adding the edge $u'r_3$. Note that for any such u' the edge $u'r_3$ is not in a triangle that we have already used. Otherwise pick any other component not yet joined to r_3 and continue. Since components have more than 2 vertices, there will be some remaining triangles of $\mathcal{T}$ meeting r_3. Add one edge from each of these to G in such a way that the degree of r_3 in G is odd. It is now clear that $G[M \cup \{r_3\}]$ is connected and r_3 has odd degree in G.

Now we shall add edges from triangles that meet r_1 and r_2 so as to make every degree in M even. Let I_j, $j = 1, 2$, be the graphs containing the edges in M of the triangles meeting r_j. Clearly each I_j is a 1-factor in M. Let $C = (u_1, u_2, \dots, u_r)$ be a component cycle of $I_1 \cup I_2$. For each vertex u_i, $i = 1, \dots, r-1$ in turn, if the degree of u_i in $G \cup \{u_r u_1\}$ is odd, add the edge $r_j u_i$ of the triangle (r_j, u_i, u_{i+1}). Otherwise add the edge $r_j u_{i+1}$ of this triangle. Now the degrees of $u_2, \dots, u_{r-1}$ are all even and the degree of u_1 is odd. If the degree of u_r is odd, add the edge $u_1 u_r$ of the triangle (r_j, u_1, u_r), otherwise add the edge $r_j u_1$. Now all the u_i have even degree and the vertices r_1 and r_2 are each joined to at least $\frac{r}{2} - 1 > 0$ vertices of M. Repeat this process for each component cycle of $I_1 \cup I_2$ in turn. The resulting graph G' has even degree at each vertex of M and contains one edge from each triangle T_i, $i \neq 1$. Also G' is connected.

Since the degree at r_3 is odd and all degrees in M are even, exactly one of the vertices r_1 or r_2 must have odd degree. Adding an edge of T_1 from this vertex to r_3 gives a connected even graph containing one edge from each triangle in $\mathcal{T}$. It therefore has an Eulerian trail meeting every vertex. This gives a good Eulerian trail of triangles in K_n.

For $n < 13$ there is at most one Steiner Triple System on K_n up to isomorphism and the result can be checked directly for $n = 1$, 3, 7, and 9 separately. For $n = 9$ the triple system can be taken as the sets of lines in $\mathbb{Z}_3 \times \mathbb{Z}_3$. If we write v_{ij} for the vertex corresponding to $(i, j) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ then the underlying trail can be taken as

$$v_{00}v_{11}v_{20}v_{12}v_{21}v_{10}v_{01}v_{11}v_{10}v_{20}v_{22}v_{02}v_{00}.$$

For $n = 7$ the triple system can be taken as triangles (v_i, v_{i+1}, v_{i+3}), $i \in \mathbb{Z}_7$, where the vertices $\{v_i : i \in \mathbb{Z}_7\}$ of K_7 are numbered mod 7. In this case the underlying trail can be taken as $(v_0, v_1, v_2, v_3, v_4, v_5, v_6)$. □

Corollary 2.3. *For any $n \neq 6, 8$ there is a packing of some graph of the form $\vec{K}_{a_1} \cdots \vec{K}_{a_r}$ into $\vec{K}_n$ with $a_i \in \{3, 4\}$ for $i < r$ and $a_r \leq 5$.*

Proof. For $n < 6$ the result is trivial since we can take $r = 1$, $a_r = n$. For $n = 7$ Lemma 2.2 gives a trail of triangles decomposing K_7. The digraph version of this decomposition gives a packing of $\vec{K}_3 \cdots \vec{K}_3$ into $\vec{K}_7$.

Now assume $n \geq 9$, $n \neq 14$. Write $n = 6m + 3 + s$ for some $m \geq 1$, $0 \leq s \leq 5$. Pick a resolvable Steiner Triple System on K_{6m+3} with classes $\mathcal{T}_1, \ldots, \mathcal{T}_{3m+1}$. Each $\mathcal{T}_i$ is a vertex disjoint collection of triangles, and K_{6m+3} is the edge disjoint union of all the triangles in $\mathcal{T} = \bigcup \mathcal{T}_i$. Let $v_1, \ldots, v_s$ be the remaining vertices of K_n. Join v_i to each triangle of $\mathcal{T}_i$ forming a collection of edge-disjoint K_4s. Now arrange the triangles of $\mathcal{T}$ into a trail of triangles using Lemma 2.2. Adding the vertices v_i gives a packing of a closed trail of K_3s and K_4s into $K_n \setminus K_s$. Splitting this closed trail at one of the K_4s and joining the K_s on the vertices $v_1, \ldots, v_s$ (if $s \geq 2$) gives the required trail. The result follows provided $s \leq 3m + 1$. This holds provided $n \neq 14$.

The only remaining case is $n = 14$. For this case consider the following Steiner Triple System on K_{13}. Label the vertices v_i with i taken mod 13 and let the set of triangles be (v_i, v_{1+i}, v_{4+i}) and (v_i, v_{2+i}, v_{7+i}) where i runs over $\mathbb{Z}_{13}$. We can arrange these into a closed trail by Lemma 2.2. The triangles (v_0, v_1, v_4), (v_5, v_7, v_{12}), (v_8, v_{10}, v_2), (v_9, v_{11}, v_3) are vertex disjoint, so joining the final vertex v of K_{14} to these gives a trail of K_3s and K_4s that misses just one edge vv_6 of K_{14}. Splitting the trail at one of the K_4s and noting that the missing edge meets this K_4 at v, we get a packing of $\vec{K}_{a_1} \cdots \vec{K}_{a_r}$ into $\vec{K}_{14}$ with $a_r = 2$, $a_{r-1} = 4$ and all other $a_i \in \{3, 4\}$. □

Proof of Theorem 1.1, assuming result for $n \leq 6$ and $n = 8$.
Assume Theorem 1.1 holds for $n \leq 6$ and $n = 8$ and assume $n \geq 7$, $n \neq 8$. Then by Corollary 2.3 we can pack a graph of the form $\vec{K}_{a_1} \cdots \vec{K}_{a_r}$ into $\vec{K}_n$ with $a_i \in \{3, 4\}$, $i < r$, and $a_r \leq 5$. It remains to pack the $\vec{C}_{m_i}$ into a graph of this form.

We can pack two $\vec{C}_3$s into $\vec{K}_3$ and four $\vec{C}_3$s into $\vec{K}_4$. By packing $\vec{K}_{a_1}$ with $\vec{C}_3$s and removing $\vec{K}_{a_1}$ we are done by induction on r unless either $r = 1$ (in which case we are done by assumption since $a_r \leq 5$) or there are at most three $\vec{C}_3$s left. If there are three $\vec{C}_3$s left, there must be some other odd m_i (since the sum of all the m_i is even). Packing four $\vec{C}_3$s into $\vec{K}_4$ gives a packing of three $\vec{C}_3$s and $\vec{C}_{m_i}$ into $\vec{K}_4 \cdot \vec{L}_{m_i-2}$. If there are fewer than three $\vec{C}_3$s left pack these as $\vec{L}_0$, $\vec{L}_3$, or $\vec{L}_{3,3}$.

Now pack the $\vec{C}_m$ with $m \neq 3$ inductively using Theorem 2.1. Assume we have a packing of $\overleftrightarrow{K}_{a_1} \cdots \overleftrightarrow{K}_{a_{j-1}} \cdot \vec{L}$ for some $\vec{L} \in \mathcal{S}$. The sum ℓ of $|\vec{L}|$ and the remaining m_i is even and $\ell \geq |E(\overleftrightarrow{K}_{a_j})|$. If $a_j \in \{3,4\}$ we can use Theorem 2.1 to pack $\vec{L}$ and some remaining $\vec{C}_{m_i}$ into $\overleftrightarrow{K}_{a_j} \cdot \vec{L}'$ for some $\vec{L}' \in \mathcal{S}$. The initial link of $\vec{L}$ is mapped to the initial link of $\overleftrightarrow{K}_{a_j} \cdot \vec{L}'$, so by composing with the packing of $\overleftrightarrow{K}_{a_1} \cdots \overleftrightarrow{K}_{a_{j-1}} \cdot \vec{L}$, we get a packing of $\overleftrightarrow{K}_{a_1} \cdots \overleftrightarrow{K}_{a_j} \cdot \vec{L}'$. Repeat this process until we run out of $\overleftrightarrow{K}_{a_j}$. If $a_r \in \{1,3,4\}$ we must have exactly used up all the cycles. If $a_r = 2$ then we must have stopped at $j = r-1$ with a packing into $\overleftrightarrow{K}_{a_1} \cdots \overleftrightarrow{K}_{a_{r-1}} \cdot \vec{L}_s$, $s = 2$ (and no more cycles) or $s = 0$ and one remaining $\vec{C}_2$. Either way we are done. If $a_r = 5$ then we have a packing into $\overleftrightarrow{K}_{a_1} \cdots \overleftrightarrow{K}_{a_{r-1}} \cdot \vec{L}$, $\vec{L} \in \mathcal{S}$ with possibly a few $\vec{C}_{m_i}$ left over. We now show that we can pack the $\vec{L}$ and these cycles into $\overleftrightarrow{K}_5$.

If $\vec{L} = \vec{L}_a$ or $\vec{L}_{a,b}$ then use Theorem 1.1 with $n = 5$, packing $\vec{C}_a$, $\vec{C}_b$, and the remaining $\vec{C}_{m_i}$ into $\overleftrightarrow{K}_5$. The closed trails in $\overleftrightarrow{K}_5$ of lengths a and b must intersect unless $(a,b) = (2,2)$, $(2,4)$, or $(2,6)$, so by permuting the vertices of $\overleftrightarrow{K}_5$ we can assume both $\vec{C}_a$ and $\vec{C}_b$ meet $\overleftrightarrow{K}_{a_{r-1}}$ and we are done. Now assume $(a,b) = (2,2)$, $(2,4)$, or $(2,6)$. The digraph $\overleftrightarrow{K}_5$ is a union of a di-star $\overleftrightarrow{K}_{1,4}$ and $\overleftrightarrow{K}_4$. Removing a $\vec{L}_{2,4}$ or a $\vec{L}_{2,6}$ (as subgraphs of the di-star) we are left packing the remaining $\vec{C}_{m_i}$ into $\overleftrightarrow{K}_4$ or $\overleftrightarrow{K}_4 \cdot \overleftrightarrow{K}_2$, both of which can be done as above. If $\vec{L} = \vec{L}_{2,2}$, remove this from the star, leaving a $\overleftrightarrow{K}_4$ with two $\vec{C}_2$s attached. Now since the number of edges in this graph is not divisible by 3, there must be some remaining $\vec{C}_{m_i}$ with $m_i \neq 3$. Pack $\vec{C}_{m_i-2}$ and all the other $\vec{C}_{m_j}$ into $\overleftrightarrow{K}_4 \cdot \overleftrightarrow{K}_2$ as above. The $\vec{C}_{m_i-2}$ must meet some vertex other than the vertex where $\overleftrightarrow{K}_4$ and $\overleftrightarrow{K}_2$ meet. Hence by adding a $\vec{C}_2$ at some other vertex we can extend the closed trail of length $m_i - 2$ to a closed trail of length m_i and we have packed a graph of the form $\overleftrightarrow{K}_2 \cdot \overleftrightarrow{K}_4 \cdot \overleftrightarrow{K}_2$ or $\overleftrightarrow{K}_4 \cdot \overleftrightarrow{K}_2 \cdot \overleftrightarrow{K}_2$. Both these graphs can be packed into the remaining edges of $\overleftrightarrow{K}_5$. □

It now remains to check the cases when $n \leq 6$ or $n = 8$. First we shall show there is no packing in the case when $n = 6$ and all $m_i = 3$.

Lemma 2.4. *There is no decomposition of $\overleftrightarrow{K}_6$ into $\vec{C}_3$s.*

Proof. Suppose we have such a decomposition and that two of the $\vec{C}_3$s form a subgraph $\overleftrightarrow{K}_3$ inside $\overleftrightarrow{K}_6$. Removing this we have a decomposition of $\overleftrightarrow{K}_6 \setminus \overleftrightarrow{K}_3$ into eight $\vec{C}_3$s. However, $\overleftrightarrow{K}_6 \setminus \overleftrightarrow{K}_3$ can be formed from a bipartite digraph $\overleftrightarrow{K}_{3,3}$ by adding six directed edges. Since each $\vec{C}_3$ must use at least one of these edges we get a contradiction.

Now pick a vertex v of $\overleftrightarrow{K}_6$. There must be exactly five $\vec{C}_3$s meeting v. On removing v we get a packing of $\overleftrightarrow{K}_5$ with five $\vec{C}_3$s and a directed graph G made up from one edge out of each $\vec{C}_3$ that meets v. Since every edge to or from v is used, G must be $(1,1)$-regular. Also G contains no $\vec{C}_2$, since that would imply that two of the $\vec{C}_3$'s form a $\overleftrightarrow{K}_3$ in the original packing. Hence G must be a directed 5-cycle. Removing G from $\overleftrightarrow{K}_5$ gives a union of a directed 5-cycle $G' = (v_0, v_1, v_2, v_3, v_4)$ and a doubly directed 5-cycle $C = (v_0, v_2, v_4, v_1, v_3)$. Pick a vertex $v_i \in V(G')$. There are exactly three $\vec{C}_3$s meeting v_i in $K_5 \setminus G$. If one of these used two edges from G' meeting v_i then the other two would use two edges from C. The union of these other two would then form a $\overleftrightarrow{K}_3$. Hence each $\vec{C}_3$ uses just one edge from G'.

Indeed, it is now clear that the five $\vec{C}_3$ are just cyclic permutations of (v_0, v_1, v_3), but the directed edges from v_i to v_{i+2} are all used twice in this packing. □

Lemma 2.5. *Assume $m_1, \dots, m_t \geq 2$ and $n \neq 6$ with $\sum_{i=1}^{t} m_i = n(n+1)$. Let $A = \sum_{m_i \text{ even}} m_i$ and $B = \sum_{i \in S} m_i$ where $|S| \in \{0, 2, 3\}$ and m_i is odd for all $i \in S$. If $A + B \geq 2n + |S|$ and Theorem 1.1 holds for $\vec{K}_n$, then we can pack $\vec{K}_{n+1}$ with closed trails of lengths $m_1, \dots, m_t$.*

Proof. If $A \geq 2n$ then take a minimal set T of values of i such that m_i is even for $i \in T$ and $A' = \sum_{i \in T} m_i \geq 2n$. Pack $\vec{C}_{m_i}$, $i \notin T$, and $\vec{C}_{A'-2n}$ (if $A' > 2n$) into $\vec{K}_n$. Now $\vec{K}_{n+1}$ is the union of $\vec{K}_n$ and a di-star $\vec{K}_{1,n}$. We can pack $\vec{C}_{m_i}$ for $i \in T$ as a union of $\frac{m_i}{2}$ $\vec{C}_2$s all meeting the same vertex $v \in V(\vec{K}_{n+1}) \setminus V(\vec{K}_n)$. If $A' > 2n$ then the last $\vec{C}_{m_i}$ is not fully packed, but we can make sure that one of the $\vec{C}_2$s used to partially pack this $\vec{C}_{m_i}$ meets the $\vec{C}_{A'-2n}$ packed inside $\vec{K}_n$ thus forming a closed trail of length m_i.

A similar argument holds if $A = 2n - 2$ and some odd m_i is at least 5. However $n(n+1) \not\equiv 2n - 2 \bmod 3$ for all n, so not all odd m_i can equal 3. Thus we are done if $A = 2n - 2$. If $A = 2n - 4$ then we can assume without loss of generality that $|S| = 2$ since only two elements of S are necessary to ensure that $A + B \geq 2n + 2$. Thus we can assume that $A \leq 2n - 2|S|$ and $|S| > 0$.

Let $T = \{i : m_i \text{ is even}\}$. Pack the cycles $\vec{C}_{m_i}$ for $i \notin S \cup T$ together with $\vec{C} = \vec{C}_{A+B-2n}$ into $\vec{K}_n$. It now remains to pack the $\vec{C}_{m_i}$ with $i \in S \cup T$ into the union of $\vec{C}$ and $\vec{K}_{1,n}$. Reduce the m_i for $i \in S$ by multiples of two to obtain $m'_i = m_i - 2k_i$, $m'_i \geq 3$, and $|\vec{C}| = \sum_{i \in S}(m'_i - 2)$. This is possible since $|S| \leq |\vec{C}| \leq B - 2|S|$, $|\vec{C}| \equiv B \bmod 2$, and all the m_i are odd for $i \in S$. Pick distinct vertices $v_1, \dots, v_{|S|} = v_0$ in $\vec{C}$ so that the part of the trail from v_i to v_{i+1} is of length $m'_i - 2$. Adding edges $\vec{vv_i}$ and $\vec{v_{i+1}v}$ will then give a packing of $\vec{C}_{m'_i}$, $i \in S$, into $\vec{C}$ and some di-edges of $\vec{K}_{1,n}$ so that each $\vec{C}_{m'_i}$ meets v. By adding other di-edges of $\vec{K}_{1,n}$ we can enlarge the closed trails until they are of length m_i, and also construct closed trails of even lengths m_i, $i \in T$. The result is a packing of $\vec{C}_{m_i}$, $i \in S \cup T$, into $\vec{K}_{n+1}$ using all the edges of $\vec{C}$ and $\vec{K}_{1,n}$.

It remains to show that we can find suitable v_i. If $|S| = 2$, say $S = \{1, 2\}$, then choose v_i arbitrarily. If $v_1 = v_2$ then shift v_i forward along $\vec{C}$ one edge to v'_1, v'_2. Then $v'_1 \neq v'_2$, since otherwise the directed edge $\vec{v_1v'_1}$ would be used twice in $\vec{C}$. Using v'_i in place of v_i now gives the result. Now assume $|S| = 3$, say $S = \{1, 2, 3\}$. We can assume we reduce m_1 and m_2 above completely before reducing m_3. Then we can assume $m_1 = m_2 = 3$, since otherwise we could have set $S = \{1, 2\}$. Hence v_1, v_2, v_3 are three consecutive vertices of $\vec{C}$. Now $|E(\vec{C})|$ is odd and $v_2 \neq v_1, v_3$. If $v_1 = v_3$ for every choice of v_i then all vertices of $\vec{C}$ are equal, which is impossible. Hence a choice of v_1 exists that makes v_1, v_2, v_3 distinct. □

Corollary 2.6. *Theorem 1.1 holds for $n \leq 6$ and $n = 8$.*

Proof. For $n < 3$ the result is clear, so assume $n \geq 3$. Let $m_1, \dots, m_t \geq 2$ and assume that r of them, $m_1, \dots, m_r$ say, are odd. Note that $\sum m_i$ is even, so r is even. If $r \leq 2$ then we can take $S = \{1, \dots, r\}$ in Lemma 2.5 and $A + B = n(n-1)$. But $n(n-1) \geq 2(n-1) + 2$ for $n \geq 3$, so we are done.

Now assume $r \geq 4$ and order the odd m_i so that $m_1 \geq \cdots \geq m_r$. Let $S = \{1,2,3\}$. Assume first that $m_i = 3$ for $3 < i \leq r$. Then $A + B \equiv n(n-1) \bmod 3$, $A + B \geq 9$, and $A + B$ is odd. If $A + B \geq 2(n-1) + 3$ we are done by Lemma 2.5. The only remaining cases are $n = 6$, $A + B = 9$, or $n = 8$, $A + B = 11$. For $n = 6$, $A + B = 9$, all the $m_i = 3$ and we do not have a packing. For $n = 8$, $A + B = 11$, there is one 2-cycle or 5-cycle and all the other $m_i = 3$. Label the vertices of K_8 as $\{u, w\} \cup \{v_i : i \in \mathbb{Z}_6\}$ and construct directed triples (u, v_i, v_{i+4}), (w, v_i, v_{i+5}), (v_i, v_{i+1}, v_{i+3}) for $i \in \mathbb{Z}_6$. It is clear that this gives a packing of $\overleftrightarrow{K}_8 \setminus \overleftrightarrow{K}_2$ with $\vec{C}_3$s. Adding the remaining $\vec{C}_2 = (u, w)$ gives the packing when there is one 2-cycle. The case when there is one 5-cycle follows by combining this $\vec{C}_2$ with one of the $\vec{C}_3$s incident to it.

If $m_4 > 3$ and $A + B < 2(n-1) + 3$ then $n = 8$, $A + B = 15$, and all $m_i \in \{3, 5\}$, By the algorithm of Lemma 2.5, we can pack all but one double edge of the $\overleftrightarrow{K}_{1,7}$ di-star and a $\vec{C}_3$ in $\overleftrightarrow{K}_7$ with the cycles that make up $A + B$ (which are all $\vec{C} + 5$s). Now $\overleftrightarrow{K}_7$ can be packed with a trail of 7 $\overleftrightarrow{K}_3$s, so after removing the closed trails already packed we get an image of a packing of $\vec{C}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_3 \cdot \overleftrightarrow{K}_2$. The total number of edges remaining is not divisible by 5. Hence there must be some $m_i = 3$ that we can pack into the $\vec{C}_3$. The remaining graph can be packed with the remaining $\vec{C}_{m_i}$ inductively using Theorem 2.1. □

We have now proved Theorem 1.1 for all n.

3. Packing dense Eulerian digraphs

We now turn to the proof of Theorem 1.2. We shall use the following powerful result of Gustavsson [4].

Theorem 3.1. *For any digraph D, there exists an $\epsilon_D > 0$ and an integer N_D, such that if G is a digraph satisfying:*

i) *$|E(G)|$ is divisible by $|E(D)|$;*

ii) *there exist non-negative integers a_{ij} such that*

$$\sum_{v_i \in V(D)} a_{ij} d_D^+(v_i) = d_G^+(u_j), \qquad \sum_{v_i \in V(D)} a_{ij} d_D^-(v_i) = d_G^-(u_j)$$

for every $u_j \in V(G)$;

iii) *if there exists $u_1\vec{u}_2 \in E(G)$ such that $u_2\vec{u}_1 \notin E(G)$ then there exists $v_1\vec{v}_2 \in E(D)$ such that $v_2\vec{v}_1 \notin E(D)$;*

iv) *$|V(G)| \geq N_D$;*

v) *$\delta^+(G), \delta^-(G) > (1 - \epsilon_D)|V(G)|$;*

then G can be written as an edge-disjoint union of copies of D.

We shall apply Theorem 3.1 with both G and D Eulerian. In this case condition ii) reduces to the requirement that each of the $d_G^+(u_j)$ is a multiple of the greatest common divisor of the $d_D^+(v_i)$, provided N_D is sufficiently large. To see that this is sufficient, write g for the greatest common divisor of the $d_D^+(v_i)$, so

$$g = \sum_{v_i \in V(D)} c_i d_D^+(v_i)$$

for some $c_i \in \mathbb{Z}$. Let $K = \sum_{v_i \in V(D)} d_D^+(v_i)/g$. For each $u_j \in V(G)$, write $d_G(u_j)/g = Kq + r$ with $0 \leq r < K$. Then we can set $a_{ij} = q + rc_i$, which will be positive provided $q \geq K \max|c_i|$. Hence a_{ij} can always be found when $d^+_G(u_j) \geq K^2 g \max|c_i|$, which is a constant depending only on D. By increasing N_D if necessary, this will hold since $d^+_G(u_j) \geq \delta^+(G) > (1-\epsilon_D)N_D$.

We shall also use the following result from [3] which is a generalization of Lemma 2.2.

Lemma 3.2. *If $\delta(H) \geq \frac{3}{4}n + \sqrt{6n} + 10$ and H has a decomposition into triangles, then these triangles can be arranged to form good Eulerian trail of triangles in H.*

Proof of Theorem 1.2.
First assume that $6 \not| \ |E(G)|$, say $|E(G)| = 6m + r$, $r \in \{2,3,4,5,7\}$. Assume that $m_1 \geq r+5$ or $m_1 \in \{r, r+3\}$. Find a closed trail of length r in G meeting some vertex v and remove these edges from G. It is clear that such a closed trail exists since $\delta^\pm$ is large. Indeed, using a greedy algorithm we can find a subgraph isomorphic to $\overleftrightarrow{K}_n$ for any $n \leq \frac{1}{2\epsilon}$, and we can take the closed trail as a subset of this $\overleftrightarrow{K}_n$. If $G = \overleftrightarrow{H}$ for some simple graph H then r is even, so we can ensure that this trail is formed from a union of $\vec{C}_2$s. Thus after removing these edges G will still be of this form. If we can pack closed trails of lengths $m_1 - r$ and m_i, $i > 1$, into the rest of the graph with the closed trail of length $m_1 - r$ meeting vertex v then we are done. Removing the closed trail of length r from G reduces $\delta^\pm(G)$ only by a constant, so by decreasing ϵ slightly, we can assume $6 \mid |E(G)|$. This reduction fails in case when all $m_i \leq r+4$. In this case, take a minimal subset of the m_i whose sum is r mod 6, pack these into G and remove. It is a simple exercise using the pigeonhole principle to show that the minimal subset must have size at most 5, and so a packing of these cycles can easily be found. Removing these cycles will not reduce the minimum degree of G much. If $G = \overleftrightarrow{H}$ we must however remove a subgraph of the same form. In this case r is even. If $r = 2$ then we can assume all $m_i \in \{3,4,6\}$. Thus there must be two $\vec{C}_4$s which we can remove as $\vec{C}_2 \cdot \vec{C}_2$s. If $r = 4$ then $m_i \in \{2,3,5,6,8\}$. If some $m_i \in \{2,8\}$ then we can remove this as a $\overleftrightarrow{K}_{1,m_i/2}$, reducing us to the $r = 2$ case. Hence we may assume all $m_i \in \{3,5,6\}$. However, in this case there must be two $\vec{C}_5$s that we can remove as a $\overleftrightarrow{K}_2 \cdot \overleftrightarrow{K}_3$. In all cases we now have a G with only slightly smaller minimum degree, $6 \mid |E(G)|$, and if the original G was of the form $\overleftrightarrow{H}$ for some simple graph H, then this still holds.

Assume $G \neq \overleftrightarrow{H}$. Define the digraph D_0 to be a closed trail of three $\vec{Q}$s. In other words, D_0 is $\vec{Q} \cdot \vec{Q} \cdot \vec{Q}$ with the initial and final links identified. If $G = \overleftrightarrow{H}$ let D_0 be a closed trail of three $\overleftrightarrow{K}_4$s. In both cases, define D as D_0 with $2d^+_{D_0}(v)$ copies of $\overleftrightarrow{K}_3$ attached to each vertex $v \in V(D_0)$. In both cases a total of 72 $\overleftrightarrow{K}_3$s have been attached, so $|E(D)| = 3(12) + 72(6) = 468 = 78(6)$.

Pack up to 77 $\overleftrightarrow{K}_3$s into G and remove so that the resulting graph G' satisfies $468 \mid |E(G')|$. Once again, such a packing clearly exists and reduces the minimum degree of G' only by a constant. Now if $N > N_D$ and $\epsilon < \epsilon_D$ we can apply Theorem 3.1 to decompose G' as an edge-disjoint union of copies of D. Note that the gcd of the degrees of D is 1 since the vertices of the $\overleftrightarrow{K}_3$s in D that don't meet D_0 have $d^+_D(v) = 2$ and there are vertices of D_0 (and hence of D) that have odd degree.

Now decompose the copies of D into their component $\overleftrightarrow{K}_3$s and D_0s. Adding back the $\overleftrightarrow{K}_3$s removed from G, we obtain a decomposition of G into D_0s and $\overleftrightarrow{K}_3$s with the property that at least $\frac{4}{5}$ of the edges leaving any vertex of G lie in $\overleftrightarrow{K}_3$s. Let G''

be the union of all the $\overleftrightarrow{K}_3$s of this decomposition. Then $\delta^+(G'') \geq \frac{4}{5}(1-\epsilon)|V(G'')|$. Thus by Lemma 3.2 the $\overleftrightarrow{K}_3$s form a good Eulerian trail of $\overleftrightarrow{K}_3$s. Each of the D_0s is a closed trail of $\vec{Q}$s or $\overleftrightarrow{K}_4$s. Since the trail of $\overleftrightarrow{K}_3$s goes through every vertex of G, we can combine it with the closed trails of $\vec{Q}$s or $\overleftrightarrow{K}_4$s to obtain a larger good Eulerian trail of graphs. Now cut this trail at v. We obtain a packing of $G_1 \cdot G_2 \cdots G_r$ into G with $G_i \in \{\overleftrightarrow{K}_3, \overleftrightarrow{K}_4, \vec{Q}\}$ and initial vertex mapped to v. If there are an odd number of m_i equal to 3, pack $\vec{L}_{3,m_1-r}$ and all the $m_i \neq 3$ into some graph of the form $G_1 \cdots G_s \cdot \vec{L}$, $\vec{L} \in \mathcal{S}$. Otherwise pack $\vec{L}_{m_1-r}$ and all the $m_i \neq 3$ into such a graph. In both cases, the total length being packed is even, and we can use Theorem 2.1 inductively as before. The only remaining cycles are an even number of $\vec{C}_3$s. Thus $6 \mid |E(\vec{L})|$ and we can assume $\vec{L} = \vec{L}_{2,4}$ or $\vec{L}_{3,3}$ or $\vec{L}_0$, since if $\vec{L}$ were any larger we could pack the next G_i. In the cases $\vec{L} = \vec{L}_{2,4}$ or $\vec{L}_{3,3}$ we can pack this and two more $\vec{C}_3$s into G_{s+1} (which must be $\vec{Q}$ or $\overleftrightarrow{K}_4$). We can now pack the remaining G_i with $\vec{C}_3$s to finish. □

4. Packing Complete Multigraphs

If we return to the undirected case, Theorem 1.1 and the results of [2] give packing results for multigraphs λK_n where λK_n is the graph on n vertices where each edge xy occurs with multiplicity λ.

Theorem 4.1. *Assume $n \geq 3$, $\sum_{i=1}^{t} m_i = \lambda\binom{n}{2}$, and $m_i \geq 2$. Then λK_n can be written as the edge-disjoint union of closed trails of lengths $m_1, \ldots, m_t$ iff either*

(a) *λ is even, or*
(b) *λ and n are both odd and $\sum_{m_i>2} m_i \geq \binom{n}{2}$.*

Before we prove Theorem 4.1 we shall need a lemma.

Lemma 4.2. *If $n > 2$ and $n \equiv 2 \bmod 3$ then $6K_n$ can be packed with triangles.*

Proof. If $n \equiv 2 \bmod 3$ then $|E(\overleftrightarrow{K}_n)| = n(n-1) \equiv 2 \bmod 3$. Hence by Theorem 1.1 we can pack $\overleftrightarrow{K}_n$ with directed triangles and a single $\vec{C}_2$. Thus by forgetting the orientations of the edges we can pack C_3s and a single C_2 into $2K_n$. Make three copies of this packing and pick three vertices v_1, v_2, v_3 of $6K_n$. By permuting the vertices in the packings we can assume the first packing has (v_1, v_2) as its C_2, the second has (v_2, v_3) and the third has (v_3, v_1). Combine these packings into a packing of $6K_n$ and replace the C_2s with two cycles (v_1, v_2, v_3). This gives a packing of $6K_n$ with triangles. □

Proof of Theorem 4.1.
The conditions are clearly necessary since if λ is odd then n must be odd for λK_n to have even degree at each vertex. Also there can be at most $\binom{n}{2}\lfloor\frac{\lambda}{2}\rfloor$ $\vec{C}_2$s, so $\sum_{m_i>2} m_i \geq \binom{n}{2}$ if λ is odd.

We now show the conditions are sufficient by induction on λ. The case $\lambda = 1$ was proved in [2]. For $\lambda = 2$, use Theorem 1.1 and forget the orientations of the edges. The special case of $2K_6$ and all $m_i = 3$ can be handled as follows. Label the vertices of K_6 as $\{v\} \cup \{v_i : i \in \mathbb{Z}_5\}$. Pack the triangles as (v, v_i, v_{i+1}) and (v_i, v_{i+1}, v_{i+3}) for $i \in \mathbb{Z}_5$. It is easy to verify that this packs 10 triangles into $2K_6$ as required.

Now assume $\lambda > 2$ and λ is even. Consider λK_n as the union of G_1 and G_2, where $\{G_1, G_2\} = \{2K_n, (\lambda-2)K_n\}$. If we could find a subset S of the m_i with

$\sum_{i\in S} m_i = |E(G_1)|$ we would be done since we could pack these into G_1 and the rest into G_2. We shall now try to find such an S.

Consider G_1 and G_2 as containers of sizes $|E(G_1)|$ and $|E(G_2)|$ respectively, into which we wish to put objects of size m_i. To fill them both we may use the following algorithm. Assume $m_1 \le m_2 \le \cdots \le m_t$. Put the m_i into the containers in order of increasing size, placing each m_i into the container that has most room left to fill at that point. Now consider what happens when we place the last object m_t. If we are lucky it will fit exactly and we are done. Otherwise it will stick out by an amount ϵ from G_1, say. If this occurs then there must be ϵ room left in the other container G_2. Since G_1 was the container with most room before we added m_t, at least ϵ of the m_t fits inside G_1, so $m_t \ge 2\epsilon$. Now assume $\epsilon \ge 2$. Split m_t as $(m_t - \epsilon) + \epsilon$ and add ϵ to G_2 and $m_t - \epsilon \ge \epsilon$ in place of m_t to G_1. Pack these lengths as closed trails by induction on λ. Now by permuting the vertices of G_1, say, we can make the packings of $C_{m_t-\epsilon}$ in G_1 and C_ϵ in G_2 meet a common vertex. Hence they combine to give a closed trail of length m_t in λK_n as desired.

Now assume $\epsilon = 1$. Since $|E(G_1)|$ is even, there must be some m_i packed inside it that is odd (their sum being $|E(G_1)| + \epsilon$). In particular $m_t \ge 3$. Replace m_t by $m_t - 1$ and replace m_i by $m_i + 1$ for some m_i in G_2. Now pack closed trails of these lengths into G_1 and G_2. Let v_1, v_2 be two vertices of G_1 that are adjacent on the closed trail of length $m_t - 1$. Let u_1, u_2 be two vertices at distance two apart on the closed trail of length $m_i + 1$. If $u_1 \ne u_2$ then by permuting the vertices of G_1, say, we can assume $v_1 = u_1$ and $v_2 = u_2$. But then swapping the v_1-v_2 and u_1-u_2 trails reconstructs closed trails of lengths m_t and m_i in λK_n from the edges of the trails of lengths $m_t - 1$ and $m_i + 1$. If $u_1 = u_2$ for all choices of u_1 and u_2, then $m_i + 1$ must be even and the closed trail of this length must alternate between just two vertices in G_2. Since $m_i + 1 > 2$, G_2 cannot be $2K_n$. Thus $G_1 = 2K_n$ and provided $m_t > 3$ there exists $v_1 \ne v_2$ at distance two apart on $m_t - 1$. Now pick u_1 and u_2 distance three apart on the closed trail of length $m_i + 1$ (which now must be distinct) and perform the same interchange of trails u_1-u_2 and v_1-v_2. Finally, if $m_t = 3$ we can assume all the m_j packed in G_2 are equal to 3. (If $m_j = 2$ for some m_j packed in G_2, swap m_j and m_t to get exact packings into G_1 and G_2). Therefore $3 \not| \, |E(G_2)|$, so $3 \not| \binom{n}{2}$. If there are at least two $m_i = 2$ assigned to G_1 then we can swap these with one of the $m_j = 3$ assigned to G_2 and get $\epsilon = 0$. If there is exactly one C_2 then $|E(G_1)| = n(n-1) \equiv 1 \bmod 3$. Since this is impossible we can assume there are no C_2s. Hence we desire a packing of C_3s into λK_n for some n with $3 \not| \binom{n}{2}$. But then $3 \mid \lambda$ and λ is even, so $6 \mid \lambda$. Also, $3 \not| \binom{n}{2}$ implies $n \equiv 2 \bmod 3$. Lemma 4.2 now gives a packing. In the case when $n = 2$ there is no such packing, hence the restriction $n \ge 3$ in the statement of the Theorem.

The case when λ is odd is similar. Split λK_n as a union of K_n and $(\lambda - 1)K_n$. We assign all the m_i which are equal to 2 to $(\lambda-1)K_n$ first. If these fill $(\lambda-1)K_n$ exactly we are done. Hence we may assume there is room left in both G_1 and G_2. Fill these up with the $m_i \ge 3$ in increasing order as above. Once again, if we do not get an exact packing then $\epsilon > 0$. As before we can split m_t if $\epsilon \ge 3$. Hence we may assume $\epsilon = 1$ or $\epsilon = 2$.

Assume $\epsilon = 2$. Then $m_t \ge 2\epsilon = 4$. As before, pack $m_t - 2$ and $m_i + 2$ and match points at distance 1 on the closed trail of length $m_t - 2$ in G_1 with points at distance 3 apart on the closed trail of length $m_i + 2$ in G_2. This fails if $G_1 = K_n$ and $m_t = 4$ (since we are not able to pack a closed trail of length $m_t - 2 = 2$ into

K_n), or if all points at distance 3 apart on the closed trail of length $m_i + 2$ in G_2 are equal. In the first case may re-assign m_t to G_2 instead since before m_t was assigned to G_1 both G_1 and G_2 had two edges left to fill. In the second case since $m_i + 2 \geq 4$ this $m_i + 2$ must be packed using just 3 vertices, going round a triangle several times. Hence $G_2 \neq K_n$ and so $G_1 = K_n$. Now we can match points 2 apart on $m_t - 2$ (distinct since $G_1 = K_n$) with those 4 apart on $m_i + 2$ (distinct since $3 \not| 4$). In all cases we can swap the v_1-v_2 and u_1-u_2 trails as before to reconstruct closed paths of lengths m_t and m_i in λK_n.

Assume $\epsilon = 1$. As before pack $m_t - 1$ and $m_i + 1$ and match points at distance 1 on $m_t - 1$ with points at distance 2 on $m_i + 1$. This fails if $G_1 = K_n$ and $m_t = 3$, or if $m_i + 1$ is even and that closed trail is packed meeting just two vertices. In the first case all the m_i assigned to G_1 are equal to 3 and $|E(G_1)| = \binom{n}{2} \equiv 2 \bmod 3$ which is impossible. In the second case $G_2 \neq K_n$, so $G_1 = K_n$ and we can find distinct points at distance 2 apart on $m_t - 1$. Then we can match them to points distance 3 apart on $m_i + 1$ and we are done as before. □

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