ON THE EDGE SPECTRUM OF SATURATED GRAPHS FOR PATHS AND STARS

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Abstract. For a given graph H, we say that a graph G on n vertices is H-saturated if H is not a subgraph of G, but for any edge $e \in E(\overline{G})$ the graph $G+e$ contains a subgraph isomorphic to H. The set of all values m for which there exists an H-saturated graph on n vertices and m edges is called the edge spectrum for H-saturated graphs.

In the present paper we study the edge spectrum for H-saturated graphs when H is a path or a star. In particular, we show that the edge spectrum for star-saturated graphs consists of all integers between the saturation number and the extremal number, and that the edge spectrum of path-saturated graphs includes all integers from the saturation number to slightly below the extremal number, but in general will include gaps just below the extremal number. We also investigate the second largest P_k-saturated graphs as well as some structural results about path-saturated graphs that have edge counts close to the extremal number.

1. Introduction

Throughout this paper all graphs will be finite, undirected, and without loops or multiple edges, and $V(G)$ and $E(G)$ will denote the vertex set and edge set, respectively, of a graph G. For a given graph H, a graph G on n vertices is called H-saturated if H is not a subgraph of G, but for any edge $e \in E(\overline{G})$ the graph $G+e$ contains a subgraph isomorphic to H. Denote by $\mathrm{SAT}(n;H)$ the set of all H-saturated graphs of order n. The saturation number $\mathrm{sat}(n;H)$ is defined as

$$\mathrm{sat}(n;H) = \min\{|E(G)| : G \in \mathrm{SAT}(n;H)\}.$$

Denote the set of saturated graphs of this minimum size by

$$\mathrm{Sat}(n;H) = \{G \in \mathrm{SAT}(n;H) : |E(G)| = \mathrm{sat}(n;H)\}.$$

Moreover, denote the extremal number by

$$\mathrm{ex}(n;H) = \max\{|E(G)| : G \in \mathrm{SAT}(n;H)\},$$

and the set of all extremal graphs by

$$\mathrm{Ex}(n;H) = \{G \in \mathrm{SAT}(n;H) : |E(G)| = \mathrm{ex}(n;H)\}.$$

The saturation number was defined by Erdős, Hajnal, and Moon in 1964 [5], and they showed that

$$\mathrm{sat}(n;K_p) = \binom{p-2}{2} + (n-p+2)(p-2)$$

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and

$$\mathrm{Sat}(n;K_p)=\{K_{p-2}+\overline{K}_{n-p+2}\}$$

for $n\ge p-2$, where $+$ denotes the join operation of graphs. In particular, they showed that $\mathrm{sat}(n;K_3)=n-1$.

Later in 1995, Barefoot, Casey, Fisher, and Fraughnaugh [3] defined the *edge spectrum for H-saturated graphs* as the set of values of m, $\mathrm{sat}(n;H)\le m\le \mathrm{ex}(n;H)$, for which there exists an H-saturated graph on n vertices and m edges. Also, they gave the following first edge spectrum result.

Theorem 1 ([3]). *Let $n\ge 5$ be an integer. There is a K_3-saturated graph with n vertices and m edges if and only if $2n-5\le m\le \frac{(n-1)^2}{4}+1$ or $m=k(n-k)$ for some positive integer k.*

In 2013, Kinnari, Faudree, Gould, and Sidorowicz [1] extended this theorem to any complete graph K_p, $p\ge 3$. Specifically, they proved the following result.

Theorem 2 ([1]). *Let $p\ge 3$ and $n\ge 3p+4$ be integers, and write $n=(p-1)q+r$ with $0\le r<p-1$.Then there is a K_p-saturated graph with n vertices and m edges graph if and only if either*

$$(p-1)\Big(n-\frac{p}{2}\Big)-2\le m\le \frac{(p-2)n^2-2n+r(r+2)-r(p-1)}{2(p-1)}+1 \tag{1}$$

or $m=|E(G)|$ for some complete $(p-1)$-partite graph G on n vertices.

Note that the range given by (1) lies strictly in the interior of $[\mathrm{sat}(n;K_p),\mathrm{ex}(n;K_p)]$ and for large n there are gaps in the K_p-saturation spectrum both near $\mathrm{sat}(n;K_p)$ and near $\mathrm{ex}(n;K_p)$.

In 2012, Gould, Tang, Wei, and Zhang [8] studied the edge spectrum of saturated graphs for small paths. In particular, they gave the edge spectrum for P_k-saturated graphs when $2\le k\le 6$. Here P_k is the path on k vertices. We state two of their results, for more details see [8].

Theorem 3 ([8]).

a) *There are P_5-saturated graphs with n vertices and m edges provided $\mathrm{sat}(n;P_5)\le m\le \mathrm{ex}(n;P_5)$, except in the cases*

$$m\in\begin{cases}\left\{\frac{3n-5}{2}\right\} & \text{if } n\equiv 3 \pmod 4,\\ \left\{\frac{3n}{2}-3,\frac{3n}{2}-2,\frac{3n}{2}-1\right\} & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

b) *For $n\ge 10$ and $(n,m)\ne(11,14)$, there are P_6-saturated graphs with n vertices and m edges provided $\mathrm{sat}(n;P_6)\le m\le \mathrm{ex}(n;P_6)$, except in the cases*

$$m\in\begin{cases}\{2n-4,2n-2,2n-1\} & \text{if } n\equiv 0 \pmod 5,\\ \{2n-4\} & \text{if } n\equiv 2,4 \pmod 5.\end{cases}$$

Note that in both cases the gaps in the spectrum, when they exist, lie close to $\mathrm{ex}(n; P_k)$. (There is a slight error in the statement given in [8] as it also claims the non-existence of P_6-saturated graphs with $n = m \in \{10, 11, 13, 14\}$. However one can form a P_6-saturated graph for all $n = m \geq 6$ by attaching pendant vertices to each of the vertices of a K_3.)

In [9], Kászonyi and Zs. Tuza established the saturation numbers for several graphs. Particularly, they proved that all smallest P_k-saturated graphs have a similar structure, defined as almost binary trees. Namely, for $k = 2p$, let T_k be a binary tree of depth $p-1$, except with root vertex of degree 3, and for $k = 2p+1$, let T_k be two binary trees of depth $p-1$ joined by an edge connecting their roots.

Theorem 4 ([9], Saturation numbers for paths)**.**

a) *For* $n \geq 3$, $\mathrm{sat}(n; P_3) = \lfloor \frac{n}{2} \rfloor$.
b) *For* $n \geq 4$,

$$\mathrm{sat}(n; P_4) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

c) *For* $n \geq 5$, $\mathrm{sat}(n; P_5) = \lceil \frac{5n-4}{6} \rceil$.
d) *Let*

$$a_k = |V(T_k)| = \begin{cases} 3 \cdot 2^{p-1} - 2 & \text{if } k = 2p, \\ 4 \cdot 2^{p-1} - 2 & \text{if } k = 2p+1. \end{cases} \tag{2}$$

Then for $n \geq a_k$ *and* $k \geq 6$, $\mathrm{sat}(n; P_k) = n - \lfloor \frac{n}{a_k} \rfloor$ *and every graph in* $\mathrm{Sat}(n; P_k)$ *consists of a forest with* $\lfloor \frac{n}{a_k} \rfloor$ *components. Furthermore, if* T *is a* P_k*-saturated tree, then* $T_k \subseteq T$.

Note that one can add any number of pendant vertices to the vertices neighboring the degree 1 vertices of T_k and obtain a tree that is still P_k-saturated.

Theorem 5 ([9])**.** *Let* $K_{1,k}$ *denote a star on* $k+1$ *vertices. Then,*

$$\mathrm{sat}(n; K_{1,k}) = \begin{cases} \binom{k}{2} + \binom{n-k}{2} & \text{if } k+1 \leq n \leq \frac{3k}{2}, \\ \lceil n\frac{(k-1)}{2} - \frac{k^2}{8} \rceil & \text{if } \frac{3k}{2} \leq n, \end{cases}$$

and

$$\mathrm{Sat}(n; K_{1,k}) = \begin{cases} \{K_k \cup K_{n-k}\} & \text{if } k+1 \leq n \leq \frac{3k}{2}, \\ \left\{G' \cup K_{\lfloor (k+1)/2 \rfloor}\right\} & \text{if } \frac{3k}{2} \leq n, \end{cases}$$

where, in the second case, G' *is any* $(k-1)$*-regular graph on* $n - \lfloor \frac{k+1}{2} \rfloor$ *vertices when either* k *is odd or* $n - \lfloor \frac{k+1}{2} \rfloor$ *is even, but when* $k-1$ *and* $n - \lfloor \frac{k+1}{2} \rfloor$ *are both odd,* G' *and* $K_{\lfloor (k+1)/2 \rfloor}$ *are joined by a single edge and every vertex of* G' *has degree* $k-1$ *in the whole graph. Furthermore, let* T *be a tree on* $k+1$ *vertices such that* $T \neq K_{1,k}$*, then* $\mathrm{sat}(n; T) < \mathrm{sat}(n; K_{1,k})$.

The corresponding extremal number for stars is simply obtained by imposing the degree condition that the maximum degree is at most $k-1$.

Theorem 6 (Folklore)**.** *For $n \geq k+1$, $\mathrm{ex}(n;K_{1,k}) = \lfloor n\frac{(k-1)}{2} \rfloor$, and the extremal graphs are*

$$\mathrm{Ex}(n;K_{1,k}) = \begin{cases} \{(k-1)\text{-regular graphs on } n \text{ vertices}\} & \text{if } n \text{ is even or } k \text{ is odd,} \\ \left\{\begin{array}{l}\text{Graphs with degree sequence} \\ k-1, k-1, ..., k-1, k-2\end{array}\right\} & \text{otherwise.}\end{cases}$$

In 1959, P. Erdős and T. Gallai [4] determined the extremal number $\mathrm{ex}(n;P_k)$ for any $k, n > 1$, as well as the corresponding extremal graphs. We state here a general version of the theorem, proved by Faudree and Schelp [7].

Theorem 7 ([7])**.** *If G is a graph with $|V(G)| = \ell(k-1)+r$, where $1 \leq r \leq k-1$, containing no paths on k vertices, then $|E(G)| \leq \ell\binom{k-1}{2} + \binom{r}{2}$ with equality if and only if G is either (i) $\left(\bigcup_{i=1}^{\ell} K_{k-1}\right) \cup K_r$; or (ii) $\left(\bigcup_{i=1}^{\ell-t-1} K_{k-1}\right) \cup (K_{(k-2)/2} + \overline{K}_{k/2+t(k-1)+r})$ for some t, $0 \leq t < \ell$, when k is even, $\ell > 0$, and $r = k/2$ or $(k-2)/2$.*

In [10], Kopylov determined the extremal number for *connected* P_k-free graphs on n vertices. Later on, in [2], Balister, Győri, Lehel, and Schelp obtained the extremal number and also gave the extremal graphs for connected P_k-free graphs on n vertices. Specifically, it was shown that the extremal graphs are of the form $G_{n,k,s} := K_s + (K_{k-2s-1} \cup \overline{K}_{n-k+s+1})$, for $k > 2s+1$. They also noted that the second class of extremal graph in the following theorem is of the form $G_{n,k,\lfloor (k-2)/2 \rfloor} = K_{(k-2)/2} + \overline{K}_{n-(k-2)/2}$ when k is even, and of the form $G_{n,k,\lfloor (k-2)/2 \rfloor} = K_{(k-3)/2} + (K_2 \cup \overline{K}_{n-(k+1)/2})$ when k is odd.

Theorem 8 ([2])**.** *Let G be a connected graph on n vertices containing no path on k vertices, $n \geq k \geq 4$. Then $|E(G)|$ is bounded by the maximum of $\binom{k-2}{2} + (n-k+2)$ and $\binom{\lceil \frac{k}{2} \rceil}{2} + \lfloor \frac{k-2}{2} \rfloor \left(n - \lceil \frac{k}{2} \rceil\right)$. If equality occurs then G is either $G_{n,k,1}$ or $G_{n,k,\lfloor (k-2)/2 \rfloor}$.*

In this paper, we show that the edge spectrum for $K_{1,k}$-saturated graphs is as large as possible for any star $K_{1,k}$.

Theorem 9. *Let n and k be two integers such that $n > k \geq 1$. Then for any integer m such that $\mathrm{sat}(n;K_{1,k}) \leq m \leq \mathrm{ex}(n;K_{1,k})$ there is a $K_{1,k}$-saturated graph on n vertices with m edges.*

We also determine that for paths P_k, the edge spectrum includes all integers from $\mathrm{sat}(n;P_k)$ to slightly below $\mathrm{ex}(n;P_k)$ when k and n are large.

Theorem 10. *Let $\varepsilon > 0$, and let k and n be integers with $k \geq k_0(\varepsilon)$ and $n \geq a_k$, where a_k is defined by (2). Then for any integer m such that $\mathrm{sat}(n;P_k) \leq m \leq \mathrm{ex}(n;P_k) - (\sqrt{2}+\varepsilon)k^{3/2}$ there exists a P_k-saturated graph on n vertices with m edges.*

We also show that $\mathrm{ex}(n;P_k) - (\sqrt{2}+o(1))k^{3/2}$ is the best possible upper bound, up to the constant $\sqrt{2}$. More precisely, we show that for each sufficiently large k there exists an infinite sequence of n and m with $\mathrm{sat}(n;P_k) \leq m \leq \mathrm{ex}(n;P_k) - \varepsilon k^{3/2}$, and no P_k-saturated graph exists with n vertices and m edges (see Corollary 15). The key to this last result is a stability theorem (Theorem 12) that gives information on the structure of connected P_k-saturated graphs that have edge counts close to the extremal number.

It remains to provide a complete description of the saturation spectrum of P_k, even when k is large. The key to a complete solution would be to understand the structure of the connected and nearly extremal graphs appearing in Theorem 12.

Finally we determine the size of the second largest P_k-saturated graphs on n vertices for all k and n.

The outline of this paper is as follows. We prove Theorem 9 in Section 2 and Theorem 10 in Section 3. Results on the structure of near-extremal P_k-saturated graphs are given in Section 4 and the size of the second largest P_k-saturated graphs are determined in Section 5.

During the preparation of this paper, we were informed that Jill Faudree, Ralph Faudree, Ronald Gould, Michael Jacobson, and Brent Thomas [6] have independently proved results on the edge spectrum of graphs, including Theorem 9 and a weaker version of Theorem 10 (with an exponential function of k in place of $(\sqrt{2}+\varepsilon)k^{3/2}$). We were also informed that Ronald Gould, Paul Horn, Michael Jacobson, and Brent Thomas proved Theorem 9 independently of the authors of this paper.

2. Edge Spectrum of $K_{1,k}$-saturated graphs

Proof of Theorem 9. The Theorem is straightforward for $k \le 3$. Thus we shall assume $k \ge 4$. We first consider the case when $k+1 \le n \le \frac{3k}{2}$. The procedure given below gives saturated graphs whose edge number ranges from $\mathrm{sat}(n; K_{1,k}) = \binom{k}{2} + \binom{n-k}{2}$ to $\mathrm{ex}(n; K_{1,k}) = \left\lfloor n\frac{(k-1)}{2}\right\rfloor$. Recall by Theorem 5 that for $k+1 \le n \le \frac{3k}{2}$ the smallest saturated graph is $K_k \cup K_{n-k}$.

The procedure can be summarized as follows: start with the smallest saturated graph $K_k \cup K_{n-k}$, pick an edge $e \in E(K_k)$, delete e from K_k, and then put two cross edges between the end points of e and vertices of K_{n-k}. Repeat this until all degrees are balanced. Of course, we must ensure that no cross edge is added twice, and no degree in the K_{n-k} ever exceeds $k-1$.

Let $w_1, \dots, w_{n-k}$ be the vertices of K_{n-k}, and let $v_1, \dots, v_k$ be the vertices of K_k. We construct partial matchings $M_1, \dots, M_{n-k}$ of K_k (and one more, called M', for odd n) as follows. If k is even, K_k has a decomposition into $k-1$ matchings $M'_1, \dots, M'_{k-1}$. If k is odd, K_{k+1} has a decomposition into k matchings. Removing a single vertex of K_{k+1} gives k partial matchings $M'_1, \dots, M'_k$ of K_k, each missing a single vertex. If n is odd, let M' be the first $\lfloor (n-k)/2 \rfloor$ edges of M'_{n-k+1}. Note that $n-k+1 \le k-1$, so M'_{n-k+1} exists. By relabeling the vertices if necessary, we may assume

$$M' = \{v_s v_{s+1} : s = 1, 3, 5, \dots, n-k-1 \text{ (or } n-k-2 \text{ if } k \text{ is even)}\}.$$

Also, when n is odd, remove the edge containing v_i (if it exists) from M'_i for $i = 1, \dots, n-k$. Now, for both n odd and n even, let M_i be an arbitrary subset of M'_i of size $k - \lceil n/2 \rceil$ for $i = 1, \dots, n-k$.

Now replace each edge in $\left(\bigcup_j M_j\right) \cup M'$ in turn by two cross edges. The edges $e = v_i v_{i'} \in M_j$ are replaced by edges $v_i w_j$ and $v_{i'} w_j$. If n is odd, the edges $v_s v_{s+1} \in M'$ are replaced by edges $v_s w_s$ and $v_{s+1} w_{s+1}$. It is clear that at each step the number of edges is increased by exactly 1 and the graph is still $K_{1,k}$-saturated. Also, no cross edge is added more than once as w_j is joined only to the endvertices of the edges in M_j and possibly to v_j when n is odd. However, all these vertices are distinct.

Continuing in this fashion and after replacing all edges in $\left(\bigcup_j M_j\right) \cup M'$, we obtain a $K_{1,k}$-saturated graph which is $(k-1)$-regular, except for one vertex w_{n-k} of degree $k-2$ when n is odd and k is even. Any such graph is extremal.

Now suppose that $\frac{3k}{2} < n$. By Theorem 5, the smallest saturated graphs are of the form $G' \cup K_t$, where $t := \left\lfloor \frac{k+1}{2} \right\rfloor$, and G' is a $(k-1)$-regular graph on $\ell := n - t$ vertices, or is regular except for one vertex of degree $k-2$ in the case when k is even and ℓ is odd, in which case we also join G' to K_t by an extra edge. We shall construct G' and partial matchings $M_1, \dots, M_t, M'$ in a manner similar to above. Note that $\ell \geq k \geq t+2$ as $n > \frac{3k}{2}$ and $k \geq 4$.

If ℓ is even, decompose K_ℓ into $\ell - 1$ edge-disjoint matchings. By taking the union of just the first $k-1$ of these matchings $M'_1, \dots, M'_{k-1}$, we obtain a $(k-1)$-regular graph. We set G' equal to this graph. If ℓ is odd, decompose K_ℓ into $(\ell-1)/2$ edge-disjoint Hamiltonian cycles $C_1, \dots, C_{(\ell-1)/2}$. Let G' be the union of the first $(k-1)/2$ of these, or the first $(k-2)/2$ of these plus a maximal matching in $C_{k/2}$ if k is even. Thus G' is $(k-1)$-regular, except for one vertex of degree $k-2$ when k is even. Decompose each cycle used into two maximal partial matchings (leaving a single edge unused). Call the partial matchings obtained $M'_1, \dots, M'_{k-1}$. In the case when k is even, M'_{k-1} will be the partial matching in $C_{k/2}$.

Thus in all cases we obtain a suitable graph G' and edge-disjoint partial matchings $M'_1, \dots, M'_{k-1}$ in G', each of size $\lfloor \ell/2 \rfloor$.

If $k-t$ is odd let M' be the first $\lfloor t/2 \rfloor$ edges of M'_{t+1}. (Recall that $t+1 \leq k-1$ and $t \leq \ell$.) By relabeling the vertices we may assume

$$M' = \{v_s v_{s+1} : s = 1, 3, 5, \dots, t-1 \text{ (or } t-2 \text{ if } k \text{ is even)}\}.$$

Then, for odd $k-t$, remove the edge meeting v_s from M'_s (if it exists) for $s = 1, \dots, t$. Now let M_i be an arbitrary subset of M'_i of size $\lfloor (k-t)/2 \rfloor$ for $i = 1, \dots, t$. Note that $\lfloor (k-t)/2 \rfloor \leq \lfloor \ell/2 \rfloor - 1$ when $k-t$ is odd, and $\lfloor (k-t)/2 \rfloor \leq \lfloor \ell/2 \rfloor$ when $k-t$ is even.

In the case when k is even and ℓ is odd, the extra edge between G' and K_t is assumed to join the unique vertex in G' of degree $k-2$ to an arbitrary vertex in K_t.

Using these partial matchings and using the same argument as before we obtain a sequence of saturated graphs with edge count starting at $\mathrm{sat}(n; K_{1,k})$ and increasing by one at each stage. In the case when G' is originally joined to K_t by a single edge e, there may be a single edge $e' \in E(G')$ that cannot be replaced by two cross edges, either because one of those cross edges would be e itself, or because the degree of the endvertex of e in K_t would be increased to k. In this case we skip e' and continue with the other edges. However, it is clear that there can be at most one skipped edge, as skipping an edge to avoid e being duplicated also ensures that the endvertex of e in K_t never exceeds $k-1$. Also, at the end of the process, all vertices in K_t have degree $k-1$ except possibly one vertex due to a skipped edge e', and the vertex v_t in the case when M' exists and k is even. These two vertices have degree $k-2$ (or $k-3$ if they are the same vertex). Hence the final graph has size at least $\mathrm{ex}(n; K_{1,k}) - 1$. Together with any extremal graph we obtain the full range and the result follows. □

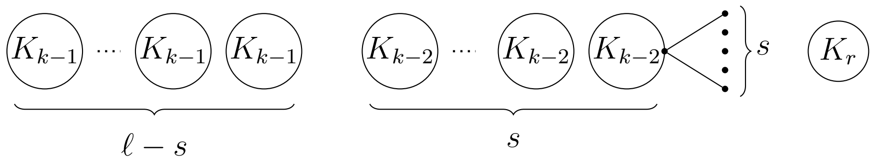

FIGURE 1. G_0

3. The Edge Spectrum of P_k-saturated graphs

In this section, we present the proof of Theorem 10. But first we prove a lemma that will be used in the proof.

Lemma 11. *Let $f(n)$ be the largest integer such that every integer between 0 and $f(n)$ can be represented as $\sum_{i\geq 1}\binom{r_i}{2}$ where $r_i \geq 0$ are integers with $\sum_{i\geq 1} r_i = n$. Then $f(n) \geq \frac{1}{2}(n-2\sqrt{n})^2$ for $n \geq 2$.*

Proof. Let $m \leq \frac{1}{2}(n-2\sqrt{n})^2$, and let $r_1 = k := \max\left\{\ell : \binom{\ell}{2} \leq m\right\}$. Then it is enough to obtain $m - \binom{k}{2} = \sum_{i\geq 2}\binom{r_i}{2}$ from the rest of the r_is, where $\sum_{i\geq 2} r_i = n-k$. Therefore, it is enough to check that $f(n-k) \geq m - \binom{k}{2}$. Since we have $m - \binom{k}{2} \leq \binom{k+1}{2} - 1 - \binom{k}{2} = k-1$, it is enough if $f(n-k) \geq k-1$ for all k such that $\binom{k}{2} \leq \frac{1}{2}(n-2\sqrt{n})^2$. As $\binom{k}{2} \geq \frac{1}{2}(k-1)^2$, we may assume $k \leq n - 2\sqrt{n}+1$. Also $n-k \geq 2\sqrt{n}-1 \geq 2$ for $n \geq 3$.

If $(n-k) - 2\sqrt{n-k} \geq \sqrt{2(k-1)}$, then we can assume by induction that

$$f(n-k) \geq \frac{1}{2}\Big((n-k) - 2\sqrt{n-k}\Big)^2 \geq k-1.$$

Therefore, it is enough to check that

$$(n-k) - 2\sqrt{n-k} \geq \sqrt{2(k-1)}$$

when $k \leq n-2\sqrt{n}+1$. Now $k-1 \leq k \leq (\sqrt{n}-1)^2$, so it is enough to show

$$(n-k) - 2\sqrt{n-k} \geq \sqrt{2}(\sqrt{n}-1) \tag{3}$$

Now $n-k \geq 2\sqrt{n}-1 \geq 1$ and $x - 2\sqrt{x}$ is an increasing function for $x \geq 1$. Thus it is enough to check that for fixed n, (3) holds for the smallest possible value of $n-k$, namely $2\sqrt{n}-1$. Hence it is enough to show

$$(2\sqrt{n}-1) - 2\sqrt{2\sqrt{n}-1} - \sqrt{2}(\sqrt{n}-1) = (2-\sqrt{2})\sqrt{n} - 2\sqrt{2\sqrt{n}-1} + (\sqrt{2}-1) \geq 0$$

Finally, we observe that this last expression is positive for $n \geq 455$. For $2 \leq n < 455$ the inequality $f(n) \geq \frac{1}{2}(n-2\sqrt{n})^2$ was verified by computer. □

Proof of Theorem 10. We split the proof into 3 parts. In Part 1, we deal with the top part of the spectrum, in Part 2 we work on the middle part of the spectrum, and in Part 3 we give the bottom part of the spectrum. Recall that k is assumed to be sufficiently large and $n \geq a_k$ where a_k is defined as in (2).

Part 1: Let $n = \ell(k-1)+r$, $1 \leq r \leq k-1$. We will deal with the cases when r is large and r is small separately.

Case 1.1: $\binom{r}{2} \geq k-2$.

Let

$$G_0 := \left(\bigcup_{i=1}^{\ell-s} K_{k-1}\right) \cup \left(\bigcup_{i=1}^{s-1} K_{k-2}\right) \cup H_s \cup K_r,$$

where $H_s = K_1 + \left(K_{k-3} \cup \overline{K}_s\right)$ (see Figure 1). Note that this graph is saturated when $r, s \geq 2$, and $|E(G_0)| = e - s(k-3)$, where $e := \mathrm{ex}(n; P_k)$. The following claim tells us that by moving vertices from some cliques to other cliques in G_0 we gain some edges.

Claim: Replacing the cliques K_{k-2} by K_{k-1-r_i} in graph G_0 so that their total order remains constant always gains $\sum \binom{r_i}{2}$ edges, where $r_i \geq 0$ are such that $\sum_{i=1}^{s-1} r_i = s-1$ is the original number of K_{k-2} cliques.

Proof of Claim: Replacing K_{k-2} by K_{k-1-r_i} increases the number of edges by

$$\begin{aligned}\tbinom{k-1-r_i}{2} - \tbinom{k-2}{2} &= \tfrac{1}{2}\big((k-1-r_i)(k-2-r_i) - (k-2)(k-3)\big) \\ &= \tfrac{1}{2}\big(r_i^2 - (2k-3)r_i + 2(k-2)\big) = (k-2)(1-r_i) + \tbinom{r_i}{2}.\end{aligned}$$

Summing over i and noting that $\sum(1-r_i) = 0$ gives the result.

By moving vertices between the K_{k-2}s, we can obtain graphs with any size between $e - s(k-3)$ and $e - s(k-3) + f(s-1)$, where f is the function defined in Lemma 11. Thus we obtain saturated graphs with any edge count in

$$\big[e - s(k-3),\ e - s(k-3) + \min\{f(s-1), k-3\}\big].$$

The bound of $k-3$ here on the number of edges added is to ensure that all r_i are sufficiently small that the graph remains P_k-saturated. Specifically, we have $k-3 < \binom{r}{2}$ so that no clique is reduced to K_{k-r-1} (which could be joined to the K_r), $k-3 < \binom{k-1}{2}$, so that no clique is reduced to an isolated vertex (which could be joined to H_s), and $k-3 < 2\binom{(k-1)/2}{2}$ so that no two cliques are reduced so that their total order is $k-1$ or less, which would allow them to be joined.

One can check that intervals obtained for all values of s from $(\sqrt[4]{2k}+2)^2$ onwards intersect consecutively. Indeed, the following two intervals

$$\big[e - s(k-3),\ e - s(k-3) + \min\{f(s-1), k-3\}\big]$$

and

$$\big[e - (s+1)(k-3), e - (s+1)(k-3) + \min\{f(s), k-3\}\big]$$

intersect if $f(s) \geq k-3$. If $s \geq s_0 := \lceil(\sqrt[4]{2k}+2)^2\rceil$, we have

$$f(s) \geq \frac{1}{2}\left(\sqrt{2k} + 4\sqrt[4]{2k} + 4 - 2\sqrt[4]{2k} - 4\right)^2 \geq k,$$

by Lemma 11. Thus the intervals intersect for $s = s_0, s_0+1, \dots, \ell$. Thus we obtain saturated graphs of all sizes in the interval

$$\big[e - \ell(k-3), e - s_0(k-3)\big].$$

Case 1.2: $\binom{r}{2} < k-2$.

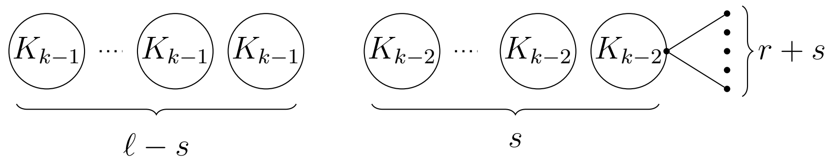

FIGURE 2. G_1

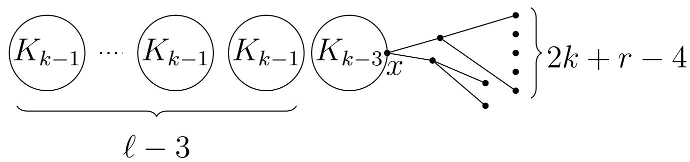

FIGURE 3. G_2

In this case we let

$$G_1 := \left(\bigcup_{i=1}^{\ell-s} K_{k-1}\right) \cup \left(\bigcup_{i=1}^{s-1} K_{k-2}\right) \cup H_{r+s},$$

where $H_{r+s} = K_1 + \left(K_{k-3} \cup \overline{K}_{r+s}\right)$ (see Figure 2). Note that this graph is saturated provided $r+s \geq 2$, and $|E(G_1)| = e - s(k-3) - a$, where $e := \text{ex}(n; P_k)$ and $a := \binom{r}{2} - r$.

We proceed as in Case 1.1, moving vertices between the K_{k-2}s so as to obtain saturated graphs with any size in the range

$$\left[e - s(k-3) - a,\ e - s(k-3) - a + \min\{f(s-1), k-3\}\right].$$

As before, the bound $k-3$ ensures that no clique is reduced so much that the graph becomes unsaturated, and these intervals overlap for $s = s_0, s_0+1, \ldots, \ell$. Thus we obtain saturated graphs of any size in

$$\left[e - \ell(k-3) - a, e - s_0(k-3) - a\right].$$

As $\binom{r}{2} < k-2$, we have $-1 \leq a = \binom{r}{2} - r \leq k-3$. So combining with Case 1.1 we obtain, in all cases, saturated graphs with any edge count in

$$\left[e - \ell(k-3) + 1, e - (s_0+1)(k-3)\right]. \tag{4}$$

Part 2: Let $n = \ell(k-1)+r$, $1 \leq r \leq k-1$. We consider the graph $G_2 := \left(\bigcup_{i=1}^{\ell-3} K_{k-1}\right) \cup H'_s$, where H'_s is K_{k-3} with a "pendant star" of size $s := 2k+r-4$ and a claw at x (see Figure 3). Note that G_2 is P_k-saturated for $s \geq 2$.

Define *Forming a Pendant Triangle* at a vertex x as follows: remove two vertices from the s pendant vertices and form a triangle whose vertices are the vertex x and the two removed vertices. By Forming a Pendant Triangle at a vertex x we gain exactly 1 edge, and the resulting graph is still P_k-saturated (Figure 4).

Using Forming a Pendant Triangle at the vertex x of G_2 repeatedly, we obtain saturated graphs with any edge counts in the interval from $|E(G_2)|$ to $|E(G_2)| + \lfloor (s-2)/2 \rfloor \geq |E(G_2)| + k - 3$.

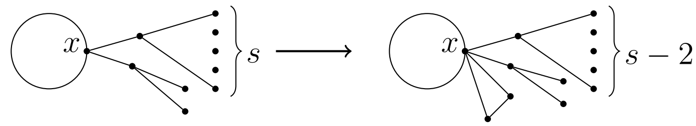

FIGURE 4. Forming a Pendant Triangle at x

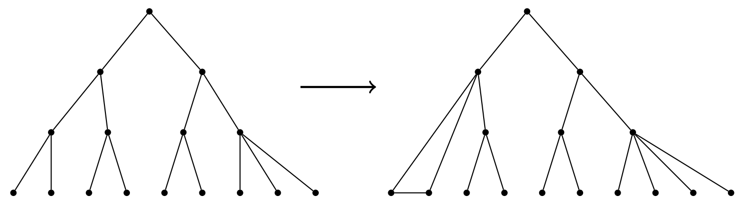

FIGURE 5. Forming a Pendant Triangle at a vertex on the bottom level

Now we move one vertex from any of the cliques in G_2 and add it to the pendant star of H'_s, obtaining a new graph G'_2 containing a component H'_{s+1}. This reduces the number of edges by $k-3$. Forming pendant triangles starting with G'_2 once again gives us saturated graphs with any edge count in the interval $|E(G'_2)|$ to $|E(G'_2)|+k-3$. This interval intersects the previous one obtained from G_2.

Repeat this process. At each step we remove a vertex from one of the cliques, adding it to H'_s as a pendent vertices. This reduces the edge count by at most $k-3$. Forming pendant triangles can then add any number of edges back up to at least $k-3$. Thus we form a sequence of intervals of numbers in the saturation spectrum whose union forms a single interval. Once a clique is reduced to 3 vertices, we move all its vertices to H'_s (which does not change the edge count) and then start removing vertices from the next clique. We continue until there are no cliques left. It is easy to see that all the graphs generated are saturated, and so the saturation spectrum contains all integers in the range

$$\left[n+\tbinom{k-3}{2}-k+3, |E(G_2)|+k-3\right].$$

However, $|E(G_2)| = e - O(k^2)$, and $\ell(k-3) \geq n/2 \geq a_k/2 \gg k^2$ so this interval intersects (4) for large k, and so we obtain all integers in the range

$$\left[n+\tbinom{k-3}{2}-k+3, e-(s_0+1)(k-3)\right]. \tag{5}$$

Part 3: We start with a smallest saturated graph G, which is a forest consisting of almost binary trees T_k in which extra pendant vertices have been added if necessary to a single vertex in the penultimate layer, and so that $|E(G)| = \mathrm{sat}(n; P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$. We can form pendant triangles in each tree component as shown in Figure 5. Indeed, a total of at least $(a_k-3)/4 \gg k^2$ pendant triangles can be formed in each tree. By forming one pendant triangle in each tree component, and then a further $\binom{k-3}{2}-k+3$ in one of them, we obtain saturated graphs with edge counts in

$$\left[\mathrm{sat}(n; P_k),\ n+\tbinom{k-3}{2}-k+3\right].$$

Hence by (5) we obtain saturated graphs with edge counts in the interval

$$\big[\mathrm{sat}(n;P_k),\mathrm{ex}(n;P_k)-(s_0+1)(k-3)\big].$$

Finally we note that $s_0=\lceil(\sqrt[4]{2k}+2)^2\rceil=(\sqrt{2}+o(1))k^{1/2}$ so $(s_0+1)(k-3)=(\sqrt{2}+o(1))k^{3/2}$ as required. □

4. Structure of large P_k-saturated graphs

In this section we study P_k-saturated graphs on n vertices that have close to $\frac{k-2}{2}n$ edges. From this we can deduce that there exist gaps in the edge spectrum for P_k-saturated graphs near the extremal number. We start by considering connected P_k-saturated graphs. The following result is somewhat stronger than is required for our purposes, however it may be of independent interest, so we state it in this strong form.

Theorem 12. *Let G be a connected P_k-saturated graph on n vertices with $|E(G)|=\frac{k-2}{2}n-\beta$, where $\beta\le k^2/128$. Then G contains a clique K_{n-t}, and the degrees of the remaining t vertices are at most $t-1$, for some $t\le 16\beta/k$.*

We note that, up to the bounds on β and t, this result is best possible. Indeed, the graph $K_{t-1}+(K_{k-2t+1}\cup\overline{K}_t)$ is a P_k-saturated graph on $n=k$ vertices with a clique of order $n-t$ and all remaining vertices are of degree $t-1$.

Proof. If $n\le k-1$ then G is a clique and we can take $t=0$ (and the condition on the remaining t vertices holds vacuously). Hence we may assume $n\ge k$. We may also assume $\beta>0$, and hence $k\ge 8$ (since $\frac{k^2}{128}\ge\beta\ge\frac12$), as by Theorem 8, $\beta=0$ implies $G=K_{k-1}$.

Suppose that G is a connected P_k-saturated graph on n vertices with $|E(G)|=\frac{k-2}{2}n-\beta$. Then by Theorem 8, we have

$$|E(G)|\le\max\left\{\tbinom{k-2}{2}+n-k+2,\tbinom{\lceil\frac{k}{2}\rceil}{2}+\left\lfloor\tfrac{k-2}{2}\right\rfloor\left(n-\left\lceil\tfrac{k}{2}\right\rceil\right)\right\}.$$

If the maximum is achieved by the second term, then $\beta\ge\lceil\frac{k}{2}\rceil\lfloor\frac{k-2}{2}\rfloor-\binom{\lceil\frac{k}{2}\rceil}{2}\ge\frac{k(k-5)}{8}>\frac{k^2}{128}$, contradicting our assumption on β. Thus we may assume $|E(G)|\le\binom{k-2}{2}+n-k+2$. Thus

$$\frac{k-2}{2}n-\beta\le\binom{k-2}{2}+n-k+2=\frac{(k-2)(k-5)}{2}+n,$$

and hence

$$n\le n_0:=\frac{(k-2)(k-5)+2\beta}{k-4}=k+\frac{2\beta-3k+10}{k-4}. \tag{6}$$

Let $t_0=\lfloor 16\beta/k\rfloor$. We let $A:=\left\{v:d_G(v)\ge\frac{n}{2}+2t_0\right\}$, and define the set of remaining vertices as $B:=\left\{v:d_G(v)<\frac{n}{2}+2t_0\right\}$.

Claim 1: $|B|\le t_0$.

Proof of Claim 1: Suppose for the contrary that $|B| > t_0$, and therefore $|A| < n - t_0$. Then we have

$$\begin{aligned} 2|E(G)| = (k-2)n - 2\beta &\leq |A|(n-1) + |B|\big(\tfrac{n}{2} + 2t_0\big) \\ &\leq (n - t_0 - 1)(n-1) + (t_0+1)\big(\tfrac{n}{2} + 2t_0\big) \\ &= n^2 - n\tfrac{t_0+3}{2} + (2t_0^2 + 3t_0 + 1). \end{aligned}$$

Using that $t_0 = \lfloor 16\beta/k \rfloor \leq k/8$ and $k \geq 8$ we deduce that $2t_0^2 + 3t_0 + 1 \leq 2\frac{k}{8} \cdot \frac{16\beta}{k} + 3\frac{k}{8} + 1 \leq 4\beta + k$ and so

$$n(k-2) \leq n^2 - \tfrac{1}{2}n(t_0+3) + 6\beta + k.$$

Now $\frac{1}{2}n(t_0+1) \geq 8\beta n/k$ and $n \geq k$, so we obtain

$$n(k-2) \leq n^2 - 2\beta\tfrac{n}{k},$$

and so, by dividing by n,

$$n \geq k - 2 + \tfrac{2\beta}{k}.$$

Combining with the upper bound (6), we obtain

$$\frac{2\beta - 3k + 10}{k-4} \geq \frac{2\beta - 2k}{k},$$

or $(2\beta - 3k + 10)k \geq (2\beta - 2k)(k-4)$. Simplifying gives $\beta \geq k(k-2)/8 \geq k^2/16$, contradicting our assumption on β.

Claim 2: A is a clique of order $n - t$ where $t = |B|$.

Proof of Claim 2: Suppose that $xy \notin E(G)$ for some pair of vertices $x, y \in A$. Then, as G is a P_k-saturated graph, $G + xy$ contains a P_k passing through the edge xy, say $P := v_1v_2...xy...v_{k-1}v_k$. If v_1v_2, respectively $v_{k-1}v_k$, lie in A then remove v_1, respectively v_k, from P. Repeat until the first and last edges of P either lie in B or join A and B. As we started with a path of length k, $|A| + |V(P) \cap B| \geq k$. We shall show that there is a path of length $|A| + |V(P) \cap B|$ in G.

Denote by $\gamma_1, \ldots, \gamma_s$ the non-trivial component paths of $P[A]$, the subgraph of P lying in A. Let $u_1, \ldots, u_w$ be any trivial components (isolated vertices) of $P[A]$. Let $t = |B|$. Then $s \leq t - 1$ and $s + w \leq t + 1$. Let the endpoints of γ_i be v_i and v_i'. Let $S = \{u_i : i = 1, \ldots, w\} \cup \{v_i, v_i' : i = 1, \ldots, s\}$.

Since all degrees of vertices in A are at least $\frac{n}{2} + 2t_0$, there are distinct vertices w_i in $A \setminus S$ such that $w_i \in N(v_i) \cap N(v_i')$ for all $i = 1, \ldots, s$. Indeed $|N(v_i) \cap N(v_i')| \geq 4t_0$, but there are at most $s + |S| + |B| = 3s + w + t < 4t_0$ common neighbors that cannot be used for w_i.

Call Q the path in G obtained from P by replacing the subpaths γ_i of P with the shorter paths $\gamma_i' := v_iw_iv_i'$. Now delete from G all the vertices of the path Q and B, and all the edges adjacent to those vertices. Exactly $r := 3s + w + t$ vertices are deleted. Thus the remaining graph has order $n - r$ and minimum degree at least $\frac{n}{2} + 2t_0 - r$. But $r < 4t_0$, so $\frac{n}{2} + 2t_0 - r > (n-r)/2$ and the remaining graph is Hamiltonian. Let C be a Hamiltonian cycle in this graph.

If Q has an endvertex in A then, as this endvertex has degree at least $\frac{n}{2} + 2t_0 > r$, it must be connected to C, in which case Q together with C will form a path of length

$|A|+|V(P)\cap B|\geq k$. If both endpoints of Q are in B, then $s>0$ as otherwise the original path P would not have used the edge xy. As the degrees of v_1 and w_1 are at least $\frac{n}{2}+2t_0$, and $4t_0>r$, they are adjacent to two consecutive vertices on C. Inserting the vertices of the Hamiltonian cycle C between v_1 and w_1 in Q gives a path containing all of $A\cup V(P)$, so again we obtain a P_k, a contradiction. Hence, A is a clique. Clearly $|A|=n-|B|=n-t$.

Claim 3: The degree of any vertex of B is at most $t-1$.

Proof of Claim 3: Pick any vertex $x\in B$. For any $y\in A$ that is not adjacent to x, the graph $G+xy$ contains a path P of length k passing through the edge xy. By removing the edge xy and all edges of P whose endpoints both lie in A, we can decompose the path into subpaths γ_i, $i=1,\dots,\ell$. These subpaths have endvertices v_i, v_i' which lie in $A\cup\{x,v,v'\}$, where v,v' are the endvertices of P. The (unique) subpath containing x will be denoted by γ_{i_0}. Note that any vertex in $A\cap V(\gamma_i)$ is only adjacent in γ_i to vertices in B.

Case 1: x is adjacent to some vertex $z\in A$ that is not on P.
Then replacing xy by xzy in P gives a P_{k+1} in G, a contradiction.

Case 2: x is adjacent to some vertex $z\in V(P)\cap A$ whose neighbors along the path P also lie in A.
If z has two neighbors u_1 and u_2 in P, replace u_1zu_2 in P by u_1u_2 and xy by xzy. If z is an endvertex of P, remove z from P and replace xy by xzy. In both cases we obtain a P_k in G avoiding xy, a contradiction.

Hence any neighbor z of x in A must lie on some γ_i. Define for each i the *discrepancy* $\delta_i:=|V(\gamma_i)\cap A\cap N(x)|-|V(\gamma_i)\cap B\setminus(N(x)\cup\{x\})|$. It is enough to show that $\sum_i\delta_i\leq 0$ as then

$$\begin{aligned}|N(x)|&=\sum_i|V(\gamma_i)\cap A\cap N(x)|+|B\cap N(x)|\\&\leq\sum_i\delta_i+|B\setminus(N(x)\cup\{x\})|+|B\cap N(x)|\\&\leq|B\setminus\{x\}|=t-1.\end{aligned}$$

Case 3: x is adjacent to an endvertex v_i of some γ_i that has both endvertices v_i,v_i' in A.
Note that in this case $i\neq i_0$. In this case remove γ_i from P, joining the neighbors $w_i,w_i'\in V(P)\setminus V(\gamma_i)$ of v_i,v_i' on P that lie outside of γ_i together to reform a path if necessary. (Both these neighbors lie in A, which is a clique.) Then replace xy with $xv_i\gamma_iv_i'y$. The result is a P_k in G, a contradiction.

Orient each γ_i so that it ends if possible with a vertex in A. If γ_i has neither endvertex in A then one of its endvertices must be x as otherwise γ_i would be the entire path P and x would not lie on P. In the case when x is an endvertex of γ_i, orient γ_i so that it starts with x.

Case 4: x is adjacent to some vertex $z\in B\cap V(\gamma_i)$ where γ_i is such that $v_i'\in A$, and either $z=v_i$ or the predecessor z^- of z on γ_i lies in A.
Write γ_i as $v_i\dots z^-z\dots v_i'$ (or $z\dots v_i'$). Remove $z\dots v_i'$ from P and join z^- (if it exists) to

the neighbor w_i' of v_i' in $P \setminus \gamma_i$ to reform a path. Note that $z^-, w_i' \in A$ when they exist. If $z = v_i$ is the start of the path γ_i then $z \in \{v, v', x\}$. But $z \neq x$ as x was assumed adjacent to z, so z is already at the end of P. Now replace xy with $xz \dots v_i'y$ to obtain a P_k in G, a contradiction.

Case 5: x is adjacent to some vertex $z \in B \cap V(\gamma_i)$ where γ_i is such that $v_i' \in B$, and either $z = v_i$ or the predecessor z^- of z on γ_i lies in A.
In this case $i = i_0$ and $v_i = x$. In particular, $z \neq v_i$. Reversing the subpath $x \dots z^-$ of P, that is, replacing $yx \dots z^-z$ with $yz^- \dots xz$, gives a P_k in G, a contradiction.

Case 6: x is adjacent to an endvertex z of γ_i, $i \neq i_0$, that lies in A, and also $v_{i_0}' \in A$.
In this case remove γ_{i_0} from P, joining its neighbors in P (which both lie in A) to reform a path if necessary. Then insert γ_{i_0} between z and its neighbor in $P \setminus \gamma_i$, giving a P_k in G.

For each γ_i, $i \neq i_0$, with one endvertex in A we have $\delta_i \leq 0$, as for each possible neighbor of x in A on γ_i there is an earlier non-neighbor in B by case 4. Indeed, we pair each element of $N(x) \cap A \cap V(\gamma_i)$ with the successor of the previous element of $A \cap V(\gamma_i)$ (or the start of γ_i if there is no previous element). Moreover, if $v_{i_0}' \in A$ then $\delta_i \leq -1$ as we then also have $v_i' \notin N(x)$ by Case 6, while v_i' is still paired with a non-neighbor of x in B.

For each γ_i with both endvertices in A we have $\delta_i \leq -1$ in general as, except for v_i, each possible neighbor of x in A on γ_i there is an earlier non-neighbor in B by Case 4, while both $v_i, v_i' \notin N(x)$ by Case 3.

If $v_{i_0}' \notin A$, then $\delta_{i_0} \leq 0$. Indeed, each neighbor of x in $A \cap V(\gamma_{i_0})$ has a successor in B that is a non-neighbor by Case 5. Hence $\sum \delta_i \leq 0$ as required.

Finally, if $v_{i_0}' \in A$ then Case 5 only implies $\delta_{i_0} \leq 1$ as the last vertex v_{i_0}' does not have a successor. However, in this case $\delta_i \leq -1$ for all $i \neq i_0$, and so $\sum \delta_i \leq 0$ as required. (We can assume that there is another γ_i as otherwise $V(P) \subseteq A \cup V(\gamma_{i_0})$ and we can form a P_k in G starting at x, traveling along γ_{i_0}, and then exhausting the vertices of A.)

The theorem now follows from Claims 1–3 together with the definition of t_0. $\square$

Lemma 13. *Let k be sufficiently large and let G be a connected P_k-saturated graph on n vertices with $|E(G)| = \frac{k-2}{2}n - \beta$. Then there exists an integer α such that $\beta = \frac{\alpha}{2}k - c\alpha^2 \geq \frac{\alpha}{32}k$ for some $c \in [0,1]$.*

Proof. We first note that if $\beta \geq (k/2)^{3/2}$ then we can simply take $\alpha = \lceil 2\beta/k \rceil$, as then $0 \leq \frac{\alpha}{2}k - \beta < \frac{k}{2} \leq \alpha^2$ and $\beta \geq \frac{\alpha}{32}k$. Thus, as k is large, we may assume $\beta \leq k^2/128$.

If $n < k$ then G is a clique. In this case write $\alpha = \min\{k-1-n, n\}$. Then $2\beta = (k-2)n - 2|E(G)| = n(k-1-n) = \alpha(k-1-\alpha) = \alpha k - \alpha(\alpha+1)$. Also, $0 \leq \alpha(\alpha+1) \leq 2\alpha^2$ for α a non-negative integer, and $\beta \geq \alpha(k-1)/2 > \alpha k/32$. Thus the result holds for $n < k$. Hence we may now assume $n \geq k$.

By Theorem 12, the graph G contains a clique of size $n-t$, where $t \leq 16\beta/k$ and rest of the vertices in G are of degree at most $t-1$. As G is connected and P_k-free, $n-t < k-1$. Write the number of edges incident with B as δt^2, so that $\delta < 1$. Write $n - t = k - 1 - \gamma$

and note that $1 \le \gamma < t$. Set $\alpha = \gamma + t$. Then

$$\begin{aligned}2\beta = (k-2)n - 2|E(G)| &= (k-2)n - (n-t)(n-t-1) - 2\delta t^2\\ &= n(k-2-n+2t+1) - t(t+1) - 2\delta t^2\\ &= n(\gamma+t) - t(t+1) - 2\delta t^2\\ &= \alpha k + (\gamma+t)(t-1-\gamma) - t(t+1) - 2\delta t^2\\ &= \alpha k - \gamma(\gamma+1) - 2t - 2\delta t^2.\end{aligned}$$

As $\gamma, t \ge 1$, $\gamma(\gamma+1) + 2t + 2\delta t^2 \le 2\gamma^2 + 4t\gamma + 2t^2 = 2\alpha^2$. Thus $\beta = \frac{\alpha}{2}k - c\alpha^2$ with $c \in [0,1]$. Also, $\alpha \le 2t \le 32\beta/k$, so $\beta \ge \frac{\alpha}{32}k$. □

Corollary 14. *Let k be sufficiently large and let G be a P_k-saturated graph on n vertices. Then there exists an integer α such that $\frac{k-2}{2}n - |E(G)| = \frac{\alpha}{2}k - c\alpha^2 \ge \frac{\alpha}{32}k$ for some $c \in [0,1]$.*

Proof. Let the components of G be G_i, $i = 1, \dots, t$, with G_i of order n_i. For each i we have $|E(G_i)| \le \frac{k-2}{2}n_i$. Thus we may apply Lemma 13 to each component, giving $\frac{k-2}{2}n_i - |E(G_i)| = \frac{\alpha_i}{2}k - c_i\alpha_i^2 \ge \frac{\alpha_i}{32}k$ for all $i \in I$. Summing over $i \in I$ gives

$$\begin{aligned}\frac{k-2}{2}n - |E(G)| &= \sum_{i=1}^{t}\Big(\frac{k-2}{2}n_i - |E(G_i)|\Big) = \sum_{i=1}^{t}\Big(\frac{\alpha_i}{2}k - c_i\alpha_i^2\Big)\\ &= \frac{k}{2}\sum_{i=1}^{t}\alpha_i - \sum_{i=1}^{t}c_i\alpha_i^2\\ &= \frac{\alpha}{2}k - c\alpha^2\end{aligned}$$

where $\alpha = \sum_{i=1}^{t}\alpha_i$, and $c \in [0,1]$ as $\sum \alpha_i^2 \le (\sum \alpha_i)^2$. Similarly $\frac{k-2}{2}n - |E(G)| \ge \frac{k}{32}\sum \alpha_i = \frac{\alpha}{32}k$. □

Corollary 15. *Let k be sufficiently large, and let $n = (k-1)\ell$. Then there is an integer $\beta_0 \sim k^{3/2}/\sqrt{8}$ such that there is no P_k-saturated graph of size* $\mathrm{ex}(n; P_k) - \beta_0$.

Proof. For $n = (k-1)\ell$, $\mathrm{ex}(n; P_k) = \frac{k-2}{2}n$. By Lemma 14, we have $|E(G)| = \frac{k-2}{2}n - \beta$ with $\beta = \frac{\alpha}{2}k - c\alpha^2 \ge \frac{\alpha}{32}k$, for some $0 \le c \le 1$. Let $\alpha_0 = \lfloor\sqrt{\frac{k-3}{2}}\rfloor$. Then for any P_k-saturated graph with $\alpha < \alpha_0$ we have

$$\beta \le \frac{\alpha_0 - 1}{2}k. \tag{7}$$

For any P_k-saturated graph with $\alpha \ge \alpha_0$ we have

$$\beta \ge \frac{\alpha_0}{2}k - \alpha_0^2. \tag{8}$$

Indeed, as α increases, so does $\frac{\alpha}{2}k - \alpha^2$ until $\alpha = k/4$, but beyond this point $\beta \ge \frac{\alpha}{32}k \ge \frac{k^2}{128} \ge \frac{\alpha_0}{2}k$, so (8) still holds. As $\alpha_0^2 < \frac{k}{2} - 1$, there is an integer $\beta_0 \sim \alpha_0 k/2 \sim k^{3/2}/\sqrt{8}$ between the bounds given by (7) and (8), and hence such that there is no P_k saturated graph with $|E(G)| = \frac{k-2}{2}n - \beta_0$. □

5. Second Largest P_k-saturated Graph

We first recall that, by Theorem 7, the extremal P_k-free graphs on $n = \ell(k-1)+r$ vertices, where $1 \le r \le k-1$, are given by either

$$G^{(c)}_{n,k} := \Big(\bigcup_{i=1}^{\ell} K_{k-1}\Big) \cup K_r,$$

or, in the special case when k is even and $r = \frac{k-2}{2}$ or $\frac{k}{2}$, can also be of the form

$$G^{(s)}_{n,k} := \Big(\bigcup_{i=1}^{\ell-t-1} K_{k-1}\Big) \cup \big[K_{(k-2)/2} + \overline{K}_{k/2+t(k-1)+r}\big],$$

for some t, $0 \le t < \ell$.

We shall now give some candidates for the second largest P_k-saturated graphs on $n = \ell(k-1)+r$ vertices. The first example consists of cliques of not-quite-maximally unbalanced orders, namely

$$H^{(c)}_{n,k} := \Big(\bigcup_{i=1}^{\ell-1} K_{k-1}\Big) \cup K_{k-2} \cup K_{r+1},$$

for $1 \le r \le k-3$ (see Figure 6). Note that $r = k-2, k-1$ are excluded as these would give extremal or unsaturated graphs respectively. The second class of candidates is

$$H^{(t)}_{n,k} := \Big(\bigcup_{i=1}^{\ell-1} K_{k-1}\Big) \cup K_{r-1} \cup \big[K_1 + (K_{k-3} \cup \overline{K}_2)\big],$$

for $r \ne 2$. The case $r = 2$ is excluded as then the graph is not P_k-saturated. The final class of candidates is

$$H^{(s)}_{n,k} := \begin{cases} \big(\bigcup_{i=1}^{\ell-1-t} K_{k-1}\big) \cup \big[K_{(k-2)/2} + \overline{K}_{k/2+(k-1)t+r}\big] & k \text{ even}, \\ \big(\bigcup_{i=1}^{\ell-1} K_{k-1}\big) \cup \big[K_{(k-3)/2} + (K_2 \cup \overline{K}_{(k-3)/2+r})\big] & k \text{ odd}, \end{cases}$$

where for k even we insist that $r \ne \frac{k-2}{2}, \frac{k}{2}$ and $0 \le t < \ell$ (see Figure 8). The restriction that $r \ne \frac{k-2}{2}, \frac{k}{2}$ is to avoid the graph becoming extremal. It should be noted that the $|E(H^{(s)}_{n,k})|$ is independent of t in the k even case, so if one of these is a second largest saturated graph then all of them are. A similar generalization in the k odd case would however lose edges as t increased, so we may assume $t = 0$ here.

Theorem 16. *For $n \ge k \ge 5$, the number of edges in the second largest P_k-saturated graph on n vertices is given by* $\max\big\{|E(H^{(c)}_{n,k})|, |E(H^{(t)}_{n,k})|, |E(H^{(s)}_{n,k})|\big\}$.

Note that in some cases more than one second largest P_k-saturated graph exists, and we do *not* guarantee that they are all of the above forms (although we believe they all are for $k \ge 8$). Some of the above graphs do not exist for some n, in which case they should simply be removed from the maximum in the theorem.

For k even, $k \ge 20$, the maximum in Theorem 16 is given by $H^{(t)}_{n,k}$ for $r = 1, k-2, k-1$; $H^{(s)}_{n,k}$ for $|r - \frac{k-3}{2}| \le \frac{1}{2}\sqrt{4k-7}$ and $r \ne \frac{k-2}{2}, \frac{k}{2}$; and $H^{(c)}_{n,k}$ for $2 \le r \le k-3$ and either $|r - \frac{k-3}{2}| \ge \frac{1}{2}\sqrt{4k-7}$ or $r = \frac{k-2}{2}, \frac{k}{2}$.

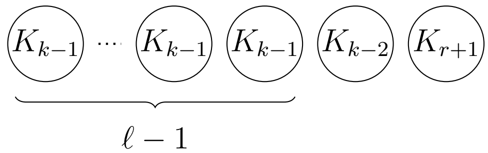

FIGURE 6. $H^{(c)}_{n,k}$

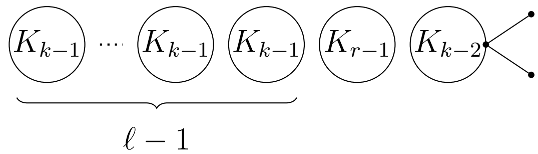

FIGURE 7. $H^{(t)}_{n,k}$

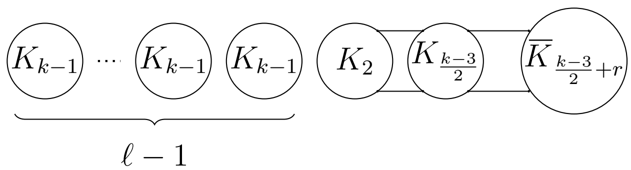

FIGURE 8. $H^{(s)}_{n,k}$ when k is odd.

For k odd, $k \geq 13$, the maximum in Theorem 16 is given by $H^{(t)}_{n,k}$ for $r = 1, k-2, k-1$; $H^{(s)}_{n,k}$ for $r = \frac{k-7}{2}, \frac{k-5}{2}, \frac{k-3}{2}, \frac{k-1}{2}$; and $H^{(c)}_{n,k}$ for $2 \leq r \leq k-3$, $r \neq \frac{k-5}{2}, \frac{k-3}{2}$.

In both cases, the pattern changes slightly for smaller values of k, although Theorem 16 still applies for any $k \geq 5$. For $k \leq 4$ the theorem fails. However, for $k \leq 3$, $\mathrm{ex}(n; P_k) = \mathrm{sat}(n; P_k)$, so there is no second largest saturated graph. For $k = 4$ there are P_4-saturated graphs of size $\mathrm{ex}(n; P_4) - 1$ for all $n \geq 4$, $n \neq 5$. These graphs are all unions of triangles and one ($n \equiv 0 \bmod 3$) or two ($n \not\equiv 0 \bmod 3$) stars $K_{1,r}$, $r \neq 0, 2$.

We start by showing that non-extremal and connected P_k-saturated graphs do not have too many edges in the case when extremal graphs exist of form $G^{(s)}_{n,k}$.

Lemma 17. *Let $n \geq k \geq 6$ and ℓ be integers such that $n = \ell(k-1) + r$, where k is even and $r = k/2$ or $(k-2)/2$. If G is a connected P_k-saturated but not extremal graph on n vertices, then $|E(G)| \leq |E(H^{(c)}_{n,k})| = \mathrm{ex}(n; P_k) - (k-2) + r$.*

The condition $k \geq 6$ is required to ensure that $r < k-2$, otherwise $H^{(c)}_{n,k}$ does not exist.

Proof. Assume for the contrary that G is a connected P_k-saturated but not extremal graph on n vertices and has more than $|E(H^{(c)}_{n,k})| = \mathrm{ex}(n; P_k) - (k-2) + r$ edges. We shall show that this leads a contradiction using a similar argument as in [2].

By a direct comparison of $H^{(c)}_{n,k}$ with the extremal graphs $G^{(c)}_{n,k-1}$ for P_{k-1} it is clear that $|E(H^{(c)}_{n,k})| \geq \mathrm{ex}(n; P_{k-1})$. Indeed, moving vertices from the K_{k-1} components to smaller cliques only decreases the number of edges. Thus we may assume that G contains a P_{k-1}. Fix a P_{k-1} in G, say $P := x_1 x_2 \ldots x_{k-1}$, and let $Y = V(G) \setminus V(P)$. For any $v \in Y$, let s_v

be the number of neighbors of v in $V(P)$, and let p_v be the number of edges in the longest path in $G[Y]$ starting at v.

By the Lemmas given in [2], we can bound $|E(G)|$ by

$$\begin{aligned}|E(G)| &\le \tbinom{k-1}{2} - \max_v f(s_v,p_v) + \sum_{v\in Y}(s_v+p_v/2),\\ &\le \tfrac{1}{n-k+1}\sum_{v\in Y}\Big(\tbinom{k-1}{2} - f(s_v,p_v) + (n-k+1)(s_v+p_v/2)\Big) \qquad (9)\end{aligned}$$

where

$$\begin{aligned} f(s,p) &:= \max\{f_0(s), f_1(s,p)\},\\ f_0(s) &:= \min\big\{\tfrac{s}{2}(2k-3s-3), s+1+\tfrac{(k-2)(k-4)}{8}\big\},\\ f_1(s,p) &:= \begin{cases}(p+1)(k-p-3)+\binom{s-1}{2}, & \text{if } s>0;\\ \frac12(k-2)\lceil \frac{p}{2}+1\rceil, & \text{if } s=0.\end{cases}\end{aligned}$$

Also $2s_v+2p_v \le k-2$ when $s_v>0$, and $p_v \le k-4$ when $s_v=0$. In particular, $2s_v+p_v\le k-2$.

Now for (9) to hold, there must be a $v\in Y$ such that

$$(\ell-1)\tbinom{k-1}{2} + \tbinom{k-2}{2} + \tbinom{r+1}{2} < |E(G)| \le \tbinom{k-1}{2} - f(s_v,p_v) + (n-k+1)(s_v+p_v/2).$$

As $(k-1)(s_v+p_v/2) \le (k-1)(k-2)/2 = \binom{k-1}{2}$, and $n=\ell(k-1)+r$, we have

$$\tbinom{k-2}{2} + \tbinom{r+1}{2} < \tbinom{k-1}{2} - f(s_v,p_v) + r(s_v+p_v/2),$$

and hence

$$f(s_v,p_v) < k-2-\tbinom{r+1}{2} + r(s_v+p_v/2).$$

For $r \ge (k-2)/2 \ge s_v+p_v/2$, the right hand side is a decreasing function of r. Thus we may substitute $r=(k-2)/2$ to obtain

$$f(s_v,p_v) < \tfrac{k-2}{8}(4s_v+2p_v-k+8). \qquad (10)$$

Claim: The only solution to the inequality (10) is $s_v = \frac{k-2}{2}$, $p_v=0$.

Proof of Claim: We first analyze the case when $s_v=0$. For $s_v=0$, the inequality implies that

$$\tfrac{k-2}{8}(2p_v-k+8) > f(0,p_v) \ge f_1(0,p_v) \ge \tfrac{k-2}{2}(\tfrac{p_v}{2}+1),$$

which implies $2p_v-k+8 > 2p_v+4$, or $k<4$, contradicting the assumption that $k\ge 6$. Hence $s_v=0$ is not a solution.

Next we analyze the case when $s_v>0$ and $p_v>0$. Note that in this case $2s_v+2p_v\le k-2$ and hence $p_v, s_v \le \frac{k-4}{2}$. We have

$$\tfrac{k-2}{8}(4s_v+2p_v-k+8) > f(s_v,p_v) \ge f_1(s_v,p_v) = (p_v+1)(k-p_v-3) + \tbinom{s_v-1}{2}.$$

Now $\frac{d}{ds}\big[\frac{(k-2)}{8}4s - \binom{s-1}{2}\big] = \frac12(k+1-2s) \ge 0$ for $s\in[1,\frac{k+1}{2}]$. As the largest value obtainable for s_v is $s_v=\frac{k-4}{2}$ when $p_v>0$, we can substitute this value for s_v. After simplifying, one obtains

$$k-14 > (p_v+1)(3k-10-4p_v).$$

As the right hand side is concave in p_v, it is enough to check it at $p_v = 1, \frac{k-4}{2}$. At $p_v = 1$ we have $k - 14 > 6k - 28$ or $k < 3$. For $p_v = \frac{k-4}{2}$ we have $k - 14 > (k-4)(k-2)$ which does not hold for any $k \geq 0$. Hence we may assume $p_v = 0$.

Now we analyze the case when $s_v > 0$, $p_v = 0$, and $\frac{s_v}{2}(2k - 3s_v - 3) \geq s_v + 1 + \frac{(k-2)(k-4)}{8}$. In this case

$$\tfrac{k-2}{8}(4s_v - k + 8) > f(s_v, 0) \geq f_0(s_v) = s_v + 1 + \tfrac{(k-2)(k-4)}{8}.$$

Simplifying gives $(k-2)(2s_v - k + 6) > 4(s_v + 1)$. Thus $2s_v > k - 6$ and so $2s_v \in \{k-4, k-2\}$. For $2s_v = k - 4$ we have $2k - 4 > 2k - 4$, a contradiction, so $s_v = (k-2)/2$.

For the case when $s_v > 0$, $p_v = 0$, and $\frac{s_v}{2}(2k - 3s_v - 3) \leq s_v + 1 + \frac{(k-2)(k-4)}{8}$, we have

$$\tfrac{k-2}{8}(4s_v - k + 8) > f(s_v, 0) \geq f_0(s_v) = \tfrac{s_v}{2}(2k - 3s_v - 3).$$

Simplifying gives $s_v(12s_v - 4k + 4) > (k-2)(k-8)$. The left hand side is concave in s_v, but at $s_v = 1$ we have $s_v(12s_v - 4k + 4) = 16 - 4k$ which is at most $(k-2)(k-8)$ for $k \geq 6$. Similarly for $s_v = (k-4)/2$ we have $s_v(12s_v - 4k + 4) = (k-4)(k-10) \leq (k-2)(k-8)$ for $k \geq 6$. Thus $s_v = (k-2)/2$ is the only solution.

Therefore, we may assume that there is a vertex $v \in Y$ with $s_v = (k-2)/2$ and $p_v = 0$. We now make few observations.

Observation 1: No vertex $u \in Y$ can be joined to an odd numbered vertex of P.

Note that v cannot be joined to adjacent vertices on P, or to the end-vertices of P as otherwise a P_k would be formed. Thus v is joined to precisely all the even numbered vertices of P. Now suppose $u \in Y$, $u \neq v$, is joined to x_i with i odd. Then the graph contains a P_k, namely $ux_ix_{i-1}\dots x_2vx_{i+1}\dots x_{k-1}$.

Observation 2: $I_1 = Y \cup \{x_i \in V(P) : i \text{ is odd}\}$ is an independent set, and there are no vertices $u \in Y$ with $s_u = 0$.

If two odd numbered vertices x_i and x_j ($i < j$) are joined, then the graph contains a P_k: $x_1x_2\dots x_ix_jx_{j-1}x_{j-2}\dots x_{i+1}vx_{j+1}x_{j+2}\dots x_{k-1}$.

If $u, w \in Y$ with $s_u > 0$, $u, w \neq v$, are joined, then there is a P_k: $x_1x_2\dots x_{i-2}vx_{k-2}x_{k-3}\dots x_iuw$, where x_i is a neighbor of u. If $w \in Y$ is joined to $u = v$, then there is a P_k: $wvx_2\dots x_{k-1}$.

Now let $S = \{u \in Y : s_u = 0\}$. Then no vertex in S can be joined to either P or $Y \setminus S$. As G is assumed to be connected, $S = \emptyset$, and I_1 is an independent set.

Finally, we complete the proof. The graph G consists of an independent set I_1 and a set of size $(k-2)/2$ comprising the even numbered vertices of P. Thus G is a subgraph of the P_k-free graph $K_{(k-2)/2} + \overline{K}_{n-(k-2)/2}$. As G is saturated, $G = K_{(k-2)/2} + \overline{K}_{n-(k-2)/2}$. But this is an extremal graph. □

Lemma 18. *Let $n \geq k \geq 6$ and ℓ be integers such that $n = \ell(k-1) + r$, where k is even, and $r = k/2$ or $(k-2)/2$. If G is a connected P_k-saturated graph on $n-1$ vertices, then* $|E(G)| \leq |E(H^{(c)}_{n,k})| = \mathrm{ex}(n; P_k) - (k-2) + r$.

Proof. From Theorem 8 we have $|E(G)| \leq |E(G_0)|$ where $G_0 = K_{(k-2)/2} + \overline{K}_{n-1-(k-2)/2}$. But $K_{(k-2)/2} + \overline{K}_{n-(k-2)/2}$ is an extremal graph, so $|E(G)| \leq \mathrm{ex}(n; P_k) - (k-2)/2 \leq \mathrm{ex}(n; P_k) - (k-2) + r$. □

Proof of Theorem 16. Assume for the contrary that there exists a non-extremal P_k-saturated graph G such that $|E(G)| > \max\{|E(H^{(c)}_{n,k}|, |E(H^{(t)}_{n,k})|, |E(H^{(s)}_{n,k})|\}$. Write $n = \ell(k-1)+r$ with $1 \le r \le k-1$, and let $C_1,\dots,C_s$ be the components of G with $|C_1| \le |C_2| \le \cdots \le |C_s|$.

Claim 1: $|C_1| + |C_2| \ge k$.

Proof of Claim 1: If $|C_1|+|C_2| < k$ we could combine C_1 and C_2 into a single clique while remaining P_k-free, thus G is not P_k-saturated.

Claim 2: $|C_s| \ge k$.

Proof of Claim 2: Indeed, if $|V(C_i)| \le k-1$ for all i, then all components are cliques. We also cannot have more components than the extremal graph $G^{(c)}_{n,k}$ without violating Claim 1, so all we can do is move vertices between the cliques of $G^{(c)}_{n,k}$. This cannot reduce the edge count when $r \in \{k-2, k-1\}$, and the second maximal edge count is achieved by $H^{(c)}_{n,k}$ in all other cases. Thus we may assume that $|C_s| \ge k$.

Claim 3: We may assume all components C_i are extremal P_k-free among the class of connected graphs of order $|C_i|$, and there are no isolated vertices.

Proof of Claim 3: Note first that all components of order at most $k-1$ are cliques and are therefore already extremal. Suppose $|C_i| \ge k$. Replace a C_i and any isolated vertex, if it exists, by a single extremal connected graph, where by 'extremal connected graph', we mean a graph that is extremal P_k-free in the class of connected graphs on a fixed number of vertices. Note that the size of an extremal connected graph is increasing in the number of vertices when there are at least k vertices. Thus we obtain a new counterexample with at least as many edges, except in two cases. The first case is when we obtain an extremal graph. However, this is only possible when k is even, $r \in \{\frac{k-2}{2}, \frac{k}{2}\}$, and all the other components are K_{k-1}s. However, Lemma 17 (when there is no isolated vertex), or Lemma 18 (when there is an isolated vertex) then imply $|E(G)| \le |E(H^{(c)}_{n,k})|$, a contradiction. The second case is when the new graph obtained is not in fact P_k-saturated. However, it is easily observed that the extremal connected graphs on at least k vertices listed in Theorem 8 can only be joined to isolated vertices while remaining P_k-free. As we have also removed any isolated vertex from G, the resulting graph is always P_k-saturated. Repeating this process for each component C_i with $|C_i| \ge k$ gives the claim.

Claim 4: We may assume G has only one component of order at least k.

Proof of Claim 4: Assume $|C_s| \ge |C_{s-1}| \ge k$. The size of the extremal connected graphs given by Theorem 8 is a convex function of the number of vertices for $n \ge k-1$. Indeed, for $n \ge k$ it is given as the maximum of two linear functions, and the extrapolation to $n = k-1$ vertices in both cases yields fewer edges than K_{k-1}. Hence moving vertices from C_{s-1} to C_s, and replacing each component by extremal connected graphs, cannot decrease the total edge count of G. Therefore, move vertices from C_{s-1} to C_s until $C_{s-1} = K_{k-1}$. This reduces the number of components of order at least k. Reorder the components and repeat until there is only one component of order at least k. The graphs produced by this process are all P_k-saturated as G contains no isolated vertices and extremal connected graphs on at least k

vertices cannot be joined to any other component while remaining P_k-free. The one special case is if we obtain an extremal graph while moving vertices from C_{s-1} to C_s. In this case we instead stop when $|C_{s-1}| = k$. For the final step to result in an extremal graph, all C_i, $i < s-1$, must be K_{k-1}s, and now $|C_s| = t(k-1)+r$ with $r = (k-2)/2$ or $(k-4)/2$. Except in the case when $k = 6$ and $r = 1$, replace C_s by an extremal graph (consisting of t cliques of order $k-1$ and one clique of order r). As all the other components must have been K_{k-1}s, and the maximal connected graph on k vertices can be taken to be $K_1 + (K_{k-3} \cup \overline{K}_2)$ (or $K_2 + \overline{K}_4$ when $k = 6$), the resulting graph is in fact $H^{(t)}_{n,k}$ (or $H^{(s)}_{n,k}$ when $k = 6$), contradicting the assumption on the size of G. For the special case $k = 6$, $r = 1$, replace $C_{s-1} = K_2 + \overline{K}_4$ and $C_s = K_2 + \overline{K}_{5t-1}$, which together have $10t + 8$ edges, with a $H^{(c)}_{5t+7,6}$ which also has $10t + 8$ edges. The graph obtained is then $H^{(c)}_{n,k}$, again contradicting the assumption on the size of G.

Claim 5: We may assume all components except C_s are K_{k-1}s.

Proof of Claim 5: Again we use the fact that the connected extremal number given by Theorem 8 is a convex function of the number of vertices for $n \geq k-1$. Also $|E(K_n)|$ is a convex function of n. Thus we can either repetitively move vertices from C_1 to C_s, or from C_s to C_1, without decreasing the number of edges.

In the first case we move all the vertices of C_1 to C_s to obtain a new counterexample with at least as many edges. The one special case is when this example is actually extremal. However, in this case we can stop when $|C_1| = 1$ and apply Lemma 18 to deduce that $|E(G)| \leq |E(H^c_{n,k})|$.

In the second case, move vertices until either $|C_s| < k$ or $|C_1| = k-1$. Suppose first that we move vertices until $|C_s| < k$. Then we are done by Claim 2, unless the graph obtained is extremal. In this case stop when $|C_s| = k$. As moving one more vertex to C_1 would result in an extremal graph, we must have all the remaining components $C_2, \ldots, C_{s-1}$ equal to K_{k-1}. As the maximal connected graph on k vertices is $K_1 + (K_{k-3} \cup \overline{K}_2)$ (or $K_2 + \overline{K}_4$ when $k = 6$), G is now of the form $H^{(t)}_{n,k}$ (or $H^{(s)}_{n,k}$ when $k = 6$).

If on the other hand we first reach a point where $|C_1| = k-1$ when moving vertices, we can again reorder the components and repeat until no components smaller than K_{k-1} remain. The only exceptional case is if when $|C_1| = k-1$ we obtain an extremal graph. In this case, stop at $|C_1| = k-2$ and replace C_s by an extremal graph (union of cliques of order $k-1$ and one clique of order either $k/2$ or $(k+2)/2$). As all the other components are K_{k-1}s, G is now of the form $H^{(c)}_{n,k}$.

To finish the proof, we may assume by the above claims that all the components except C_s are K_{k-1}s and that C_s is one of the extremal connected graphs given by Theorem 8 with $|C_s| \geq k$.

Case 1: C_s is of the first form in Theorem 8, namely $K_1 + (K_{k-3} + \overline{K}_{r+1})$, $r \geq 1$.
As noted in [2], this occurs only when $k-1+r \leq (5k-7)/4$, or $r \leq (k-3)/4$. In particular, $r < k-3$. Now $|E(G)| = (\ell-1)\binom{k-1}{2} + \binom{k-2}{2} + r + 1$. Comparing with the graph $H^{(c)}_{n,k}$

which has $(\ell-1)\binom{k-1}{2}+\binom{k-2}{2}+\binom{r+1}{2}$ edges we see that either $r=1$ and G is $H_{n,k}^{(t)}$, or $|E(G)|\le|E(H_{n,k}^{(c)})|$.

Case 2: C_s is of the second form in Theorem 8.
In this case either $G=H_{n,k}^{(s)}$ or, in the case when k is odd, G may be of the form $\left(\bigcup_{i=1}^t K_{k-1}\right)\cup H_{n-(k-1)t}^{(s)}$ which has strictly fewer edges for $t>0$. □

References

[1] K. Amin, J. Faudree, R.J. Gould, and E. Sidorowicz, On the Non-$(p-1)$-Partite K_p-Free Graphs, *Discussiones Mathematicae Graph Theory,* **33** (1) (2013) 9–23.

[2] P.N. Balister, E. Győri, J. Lehel, and R.H. Schelp, Connected graphs without long paths, *Discrete Math.* **308** (2008) 4487–4494.

[3] C. Barefoot, K. Casey, D. Fisher, and K. Fraughnaugh, Size in maximal triangle-free graphs and minimal graphs of diamerter 2, *Discrete Math.* **138** (1995) 93–99.

[4] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.,* **10** (1959) 337–356.

[5] P. Erdős, A. Hajnal, and J.W. Moon, A problem in graph theory, *Amer. Math Monthly* **71** (1964) 1107–1110.

[6] J. Faudree, R. Faudree, R. Gould, M. Jacobson, and B. Thomas, Saturation spectrum of Paths and Stars, *Discussiones Mathematicae Graph Theory* (2017) (in press).

[7] R.J. Faudree and R.H. Schelp, Paths Ramsey numbers in multicolorings, *Journal of Combin. Theory* B **19** (1975) 150–160.

[8] R.J. Gould, W. Tang, E. Wei, C.-Q. Zhang, The edge spectrum of the saturation number for small paths, *Discrete Math.* **312** (2012) 2682–2689.

[9] L. Kászonyi and Zs. Tuza, Saturated graphs with minimal number of edges, *J. Graph Theory* **10** (2)(1986) 203–210.

[10] G.N. Kopylov, On maximal paths and cycles in a graph, *Soviet Math. Dokl.* Vol **18** (1977) 593–596.

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