Flat Littlewood Polynomials Exist

Paul Balister[1] Béla Bollobás[1,2] Robert Morris[3]
Julian Sahasrabudhe[2] Marius Tiba[2]

[1]University of Memphis

[2]University of Cambridge, UK

[3]IMPA, Rio, Brazil

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Littlewood Polynomials

We say that a polynomial $P(z)$ of degree $n-1$ is a *Littlewood polynomial* if

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Such polynomials have been studied extensively for more than 100 years, starting with the work of Hardy and Littlewood on Diophantine approximations in 1916.

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$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 \, d\theta = \sum_{k=0}^{n-1} |\varepsilon_k|^2 = n,$$

and so there must exist points $z \in S^1$ where

$$|P(z)| \geq \sqrt{n}$$

and also points $z \in S^1$ where

$$|P(z)| \leq \sqrt{n}.$$

Flat Littlewood Polynomials

In 1957 Erdős asked whether it is possible to choose Littlewood polynomials so that

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In 1966 Littlewood made a series of conjectures about Littlewood polynomials. Possibly the most famous being that such 'Flat' polynomials do exist.

Conjecture (Littlewood's Flat Polynomial Conjecture (1966))

There exists absolute constants $\Delta > \delta > 0$ such that for infinitely many n there exists a Littlewood polynomial of degree $n-1$ with

$$\delta\sqrt{n} \le |P(z)| \le \Delta\sqrt{n}, \qquad \text{for all } z \in S^1.$$

Related questions

If we relax the condition that $\varepsilon_k \in \{\pm 1\}$ and instead only insist that $|\varepsilon_k| = 1$, $\varepsilon_k \in \mathbb{C}$, then flat polynomials *do* exist.

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Indeed Kahane (1980) showed that 'ultra-flat' polynomials $P(z)$ exist:

$$|P(z)| = (1 + o(1))\sqrt{n}, \qquad \text{for all } z \in S^1.$$

Bombieri and Bourgain (2009) improved the methods of Kahane to gave an effective construction of ultra-flat polynomials with even tighter bounds $|P(z)| = \sqrt{n} + O(n^{7/18+o(1)})$.

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Note that a *Random* Littlewood polynomial will typically only be bounded by $\sqrt{n \log n}$.

Polynomials with good upper bounds

In the 1950s, Shapiro and Rudin independently gave the following construction of Littlewood polynomials of degree $n-1=2^t-1$.

$$P_0(z) = Q_0(z) = 1$$
$$P_{t+1}(z) = P_t(z) + z^{2^t} Q_t(z)$$
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So there is an infinite sequence of polynomials satisfying the upper bound with $\Delta = \sqrt{2}$.

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Theorem (B., (2019))

$$|P_{<n}(z)| \le \sqrt{6n}, \qquad \textit{for } z \in S^1 \textit{ and any } n \ge 0.$$

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Since then, several other constructions of polynomials have been given that satisfy the upper bound $|P(z)| \le \Delta\sqrt{n}$ for some constant Δ.

Rudin–Shapiro polynomials

Unfortunately, the Rudin–Shapiro polynomials *don't* satisfy the lower bound. Indeed, Rodgers (2012) showed that $P_t(z)/\sqrt{2^t}$ is asymptotically uniformly distributed in the disk $|z| \leq \sqrt{2}$ for large t when z is uniformly distributed in S^1

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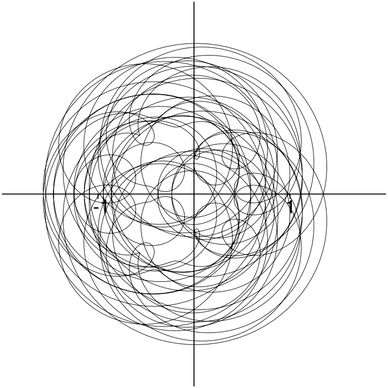

Plot of $P_6(z)/\sqrt{2^6}$, $z \in S^1$.

Polynomials with good lower bounds

In terms of lower bounds, the best result was a construction by Carroll, Eustice and Figiel (1977), that there exist Littlewood polynomials with

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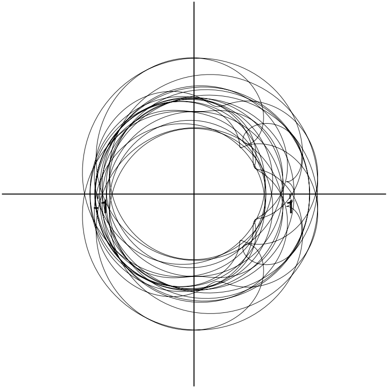
The proof is a simple recursive construction based on a specific very good example with $n = 13$ (based on a Barker sequence), and the observation that if $P(z)$ is a Littlewood polynomial of degree $n-1$ then $P(z)P(z^n)\ldots P(z^{k-1})$ is a Littlewood polynomial of degree $n^k - 1$.

Numerical searches

Exhaustive search for small values of n by Odlyzko (2018) suggest that Flat Littlewood Polynomials exist, but that ultra flat Littlewood polynomials do not.



Plot of 'optimal' $P(z)/\sqrt{n}$ for $n = 45$.

The main result

It is enough to prove the result for sufficiently large n as there exist Littlewood polynomials of any degree ≥ 2 with no roots on S^1. Also, multiplying by z^{-k} has no effect on $\sup|P(z)|$, and adding or removing a bounded number of terms only affects δ, Δ slightly. Thus it is enough to show:

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Theorem (B, Bollobás, Morris, Sahasrabudhe, Tiba (2019))

For every sufficiently large $n \in \mathbb{N}$, there exists a Littlewood 'polynomial' $P(z) = \sum_{k=-2n}^{2n} \varepsilon_k z^k$ such that

$$2^{-160}\sqrt{n} \leq |P(z)| \leq 2^{12}\sqrt{n}$$

for all $z \in \mathbb{C}$ with $|z| = 1$.

Proof Strategy

Choose a set $C \subseteq [2n] = \{1, \ldots, 2n\}$.
Set $\varepsilon_{-k} = \varepsilon_k$ for each $k \in C$.
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Setting $z = e^{i\theta}$, the polynomial $P(z)$ then decomposes as

$$\sum_{k=-2n}^{2n} \varepsilon_k z^k = \varepsilon_0 + 2 \sum_{k \in C} \varepsilon_k \cos(k\theta) + 2i \sum_{k \in S} \varepsilon_k \sin(k\theta) = \varepsilon_0 + c(\theta) + is(\theta)$$

The real part of this expression is a cosine polynomial, while the imaginary part is a sine polynomial.

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We then aim to choose $s(\theta) = O(\sqrt{n})$, and with $s(\theta)$ large on the intervals where $c(\theta)$ is small.

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- 'aligned': the endpoints of each interval in $\mathcal{I}$ lie in $\frac{\pi}{n}\mathbb{Z}$;

The cosine polynomial $c(\theta)$

We prove the following.

Theorem

There exists a cosine polynomial

$$c(\theta) = \sum_{k \in C} \varepsilon_k \cos(k\theta),$$

with $\varepsilon_k \in \{-1, 1\}$ *for every* $k \in C$, *and a set* $\mathcal{I}$ *of disjoint intervals as above such that*

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for all $\theta \notin \bigcup_{I \in \mathcal{I}} I$, *and* $|c(\theta)| \leq \sqrt{n}$ *for all* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

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Proof is explicit construction.

Construction of $c(\theta)$

Let $n \in \mathbb{N}$ be sufficiently large, choose $2^{-43} < \gamma \le 2^{-40}$ such that

$$\gamma n = 2^{t+11} + 2^t - 1$$

for some *odd* integer t.

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Define C by setting $C = 2C'$, where

$$C' := \{2^{t+10}, \ldots, 2^{t+10} + 2^t - 1\} \cup \{2^{t+11}, \ldots, 2^{t+11} + 2^t - 1\},$$

so that C is a set of *even* integers.

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They have the right symmetry by the fact that C only contains even integers.

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Hence $c(\theta)$, $c'(\theta)$, $c''(\theta)$, $c'''(\theta)$ can't all be small, and none of these vary much over a distance $O(1/n)$.

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Explicit calculation shows $c(\theta)$ large near 0 and $\pi/2$.

The idea is that as $|P_t|^2 + |Q_t|^2 = 2^{t+1}$, we cannot have both P_t and Q_t small.

The z^T and z^{2T} vary much faster than P_t and Q_t.

Expressions such as $f(x) = a_1 \cos(x + \alpha_1) + a_2 \cos(2x + \alpha_2)$ cannot have $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ simultaneously small if $|a_1|^2 + |a_2|^2$ large.

Hence $c(\theta)$, $c'(\theta)$, $c''(\theta)$, $c'''(\theta)$ can't all be small, and none of these vary much over a distance $O(1/n)$.

Hence $c(\theta)$ cannot be small on a large interval.

The sine polynomials

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- An *even* sine polynomial $s_e(\theta)$, made up of $\pm\sin k\theta$ terms with k even $k \notin C$.
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We just want the even sine polynomial to be small everywhere, so define

$$s_e(\theta) = \mathrm{Im}(P(z))$$

where $z = e^{2i\theta}$ and $P(z)$ is made up from a Rudin–Shapiro polynomial by only including terms z^k, $k \leq 2n$ and $k \notin C$.

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Theorem

$|s_e(\theta)| \leq 6\sqrt{n}$.

The odd sine polynomial $s_o(\theta)$

Strategy: for each 'bad' interval $I \in \mathcal{I}$ on which $c(\theta)$ is small, we will choose a direction (positive or negative), and attempt to 'push' the sine polynomial in that direction on that interval.

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Strategy: for each 'bad' interval $I \in \mathcal{I}$ on which $c(\theta)$ is small, we will choose a direction (positive or negative), and attempt to 'push' the sine polynomial in that direction on that interval.

In other words, we pick a step function that is $\pm K\sqrt{n}$ on each of the bad intervals, and zero elsewhere, where $K = 2^7$ is a large constant. We then attempt to approximate this step function with a sine polynomial, the hope being that we can do so with an error of size $O(\sqrt{n})$ on each bad interval (much smaller than $K\sqrt{n}$).

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In order to carry out this plan we will use an old result of Spencer on combinatorial discrepancy, first to choose the step function so that its Fourier coefficients are small, and then to show we can approximate it sufficiently closely by some $s_o(\theta)$.

Combinatorial Discrepancy

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We use a variant of Spencer's theorem that was proved by Lovett and Meka (2015), who also gave a beautiful polynomial-time randomized algorithm for finding a coloring with small discrepancy.

This result provides a 'partial coloring' result that can easily be extended to the following 'full coloring' theorem.

Theorem

Let $v_1, \ldots, v_m \in \mathbb{R}^n$ *and* $x_0 \in [-1,1]^n$. *If* $c_1, \ldots, c_m \geq 0$ *are such that*

$$\sum_{j=1}^{m} \exp\left(-c_j^2/14^2\right) \leq \frac{n}{16},$$

then there exists an $x \in \{-1,1\}^n$ *such that*

$$|\langle x - x_0, v_j \rangle| \leq (c_j + 30)\sqrt{n} \cdot \|v_j\|_\infty$$

for every $j \in [m]$.

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Write $\tilde{s}_o(\theta) = \frac{K\pi}{2}\sqrt{n}\sum \alpha_I 1[\theta \in I]$ with $\alpha_I \in \{\pm 1\}$ (α_I having appropriate symmetries under $\theta \mapsto \pi \pm \theta$).

Then by Fourier inversion $\tilde{s}_o(\theta) \approx \hat{s}_o(\theta) := \sum \hat{\varepsilon}_k \sin(k\theta)$ where

$$\hat{\varepsilon}_k = K\sqrt{n} \sum_{I \subseteq [0,\pi/2]} \alpha_I \int_I \sin(k\theta)\, d\theta$$

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is a linear combination of the α_I.

As the bad intervals $I \in \mathcal{I}$ are 'few' and 'small', we typically we would expect $\hat{\varepsilon}_k$ to be small. Unfortunately a random choice of α_I would occasionally result in a large $\hat{\varepsilon}_k$.

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By using the combinatorial discrepancy theorem, we show (using the 'few' and 'small' conditions) that

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There exists a choice of $\alpha_I \in \{\pm 1\}$ *such that all* $\hat{\varepsilon}_k \in [-1, 1]$.

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Theorem

$|\hat{s}_o(\theta)| \geq \frac{2K}{3}\sqrt{n}$ *for* $\theta \in I \in \mathcal{I}$, *and* $|\hat{s}_o(\theta)| \leq 5K\sqrt{n}$ *everywhere.*

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Proof uses that fact that the endpoints of I are aligned to multiples of π/n (and the distance between them is at least π/n). This makes the errors caused by truncating the Fourier series largely cancel out.

Choosing the odd sine polynomial

Theorem

There exists a choice of $\varepsilon_k \in \{\pm 1\}$ *such that*

$$|s_o(\theta) - \hat{s}_o(\theta)| = \Big| \sum_k (\varepsilon_k - \hat{\varepsilon}_k) \sin(k\theta) \Big| \leq 72\sqrt{n}$$

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Proof uses the combinatorial discrepancy result to bound the values and kth derivatives $|s_o^{(k)}(\theta) - \hat{s}_o^{(k)}(\theta)|$ at a finite number of points $\theta = \theta_i$.

Fortunately the bounds get weaker as k gets larger, and our version of the combinatorial discrepancy theorem is enough to bound the derivatives sufficiently well that Taylor's theorem is able to then bound $|s_o(\theta) - \hat{s}_o(\theta)|$ for *all* $\theta \in [0, 2\pi]$.

The odd sine polynomial $s_o(\theta)$

We deduce that:

Theorem

Let $\mathcal{I}$ be a collection of intervals satisfying the properties defined above. Then there exists a sine polynomial

$$s_o(\theta) = \sum_{k \in S_o} \varepsilon_k \sin(k\theta),$$

with $\varepsilon_k \in \{-1, 1\}$ for every $k \in S_o$, such that

(*i*) $|s_o(\theta)| \geq 10\sqrt{n}$ *for all* $\theta \in I \in \mathcal{I}$, *and*

(*ii*) $|s_o(\theta)| \leq 2^{10}\sqrt{n}$ *for all* $\theta \in \mathbb{R}$.

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To deduce our main Theorem, we simply set

$$P(e^{i\theta}) := \big(1 + 2c(\theta)\big) + 2i\big(s_e(\theta) + s_o(\theta)\big),$$

and combine the above theorems.

And so . . .

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Flat Littlewood polynomials exist!

And so ...

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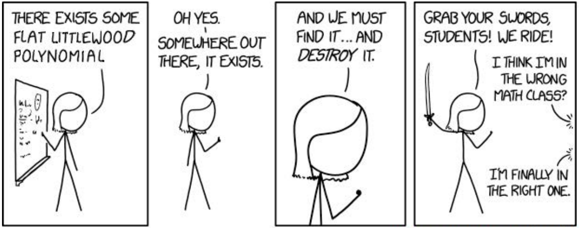

(Adapted from xkcd.com/1856/)