On Erdős Covering Systems

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Joint work with Béla Bollobás, Rob Morris, Julian Sahasrabudhe, and Marius Tiba.

Covering Systems

A **covering system** is a finite collection of arithmetic progressions

$$(a_i \bmod d_i) := a_i + d_i\mathbb{Z}, \qquad i = 1, \ldots, k,$$

that cover $\mathbb{Z}$:

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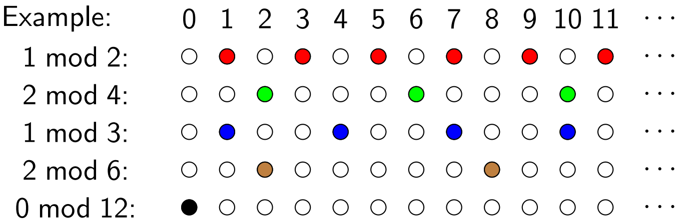
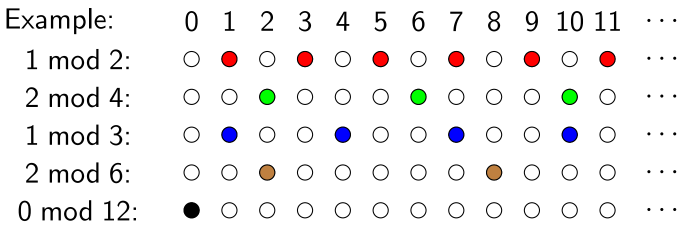
We are interested in the case when the moduli d_i are **distinct**, say

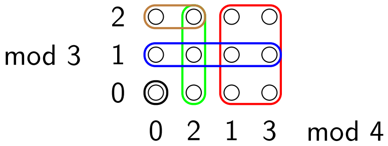
$$1 < d_1 < d_2 < \cdots < d_k.$$

Covering Systems





Or using the CRT: $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$



Some Questions

Question (Erdős, 1950)

If all the moduli d_i are distinct, can the smallest one $d_1 = \min d_i$ be arbitrarily large?

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We give a *much* simpler proof of this result, improving it to:

Theorem (BBMST, 2018)

If all the moduli d_i are distinct, $d_1 < 616000$.

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The methods used are complex and highly optimized, and are only *just* enough for it to work.

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Our techniques give a very simple proof of this, and strengthens it to:

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This last result is much harder to prove.

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In any covering system with there must exists $i \neq j$ with $d_i \mid d_j$.

Setup for the proofs

Write $Q = \mathrm{lcm}\{d_i\} = p_1^{e_1} \dots p_n^{e_n}$. We can think of a covering system as a cover of the hypercuboid

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We identify subsets

$$S \subseteq \mathbb{Z}_{Q_i} = \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_i^{e_i}}$$

with the subset

$$S \times \mathbb{Z}_{p_{i+1}^{e_{i+1}}} \subseteq \mathbb{Z}_{Q_{i+1}}$$

or

$$S \times \mathbb{Z}_{p_{i+1}^{e_{i+1}}} \times \dots \mathbb{Z}_{p_i^{e_i}} \subseteq \mathbb{Z}_Q$$

and we identify arithmetic progressions ($a_j \bmod d_j$), $d_j \mid Q_i$, with the corresponding subset of $\mathbb{Z}_{Q_i}$.

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$R_i = \mathbb{Z}_Q \setminus \bigcup_{d_j \in D_i} (a_j \bmod d_j) = R_{i-1} \setminus B_i$
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We construct a sequence of probability measures $\mathbb{P}_i$ on $\mathbb{Z}_Q = \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_n^{e_n}}$ which is uniform on each fibre of $x \in \mathbb{Z}_{Q_i}$, i.e., it is a product of a (non-trivial) measure on $\mathbb{Z}_{Q_i}$ (which we also call $\mathbb{P}_i$) with the uniform measure on $\mathbb{Z}_{Q/Q_i}$.

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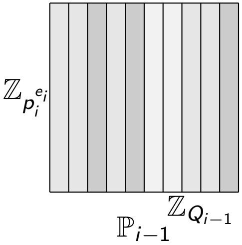

$\mathbb{P}_{i-1}$ is defined on $\mathbb{Z}_{Q_{i-1}}$ and extended uniformly on each fibre $\{x\} \times \mathbb{Z}_{p_i^{e_i}} \subseteq \mathbb{Z}_{Q_{i-1}} \times \mathbb{Z}_{p_i^{e_i}} = \mathbb{Z}_{Q_i}$.

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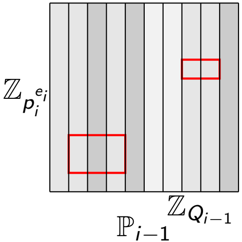

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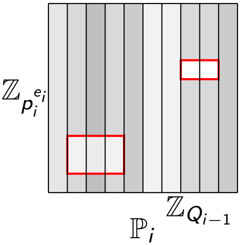

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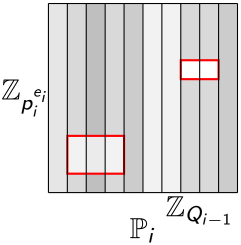

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But not so much that it increases the density by more than $1/(1-\delta_i)$ anywhere, where $\delta_i \in (0, 1/2]$ is an appropriately chosen constant. Note that if less than δ_i of the fibre is removed, then no measure is placed inside B_i in that fibre.

Setup for the proofs

Formally, for each $x \in \mathbb{Z}_{Q_{i-1}}$, define

$$\alpha_i(x) = \frac{\mathbb{P}_{i-1}\big(x \cap B_i\big)}{\mathbb{P}_{i-1}(x)} = \frac{\Big|\Big\{y \in \mathbb{Z}_{p_i^{e_i}} : (x,y) \in B_i\Big\}\Big|}{p_i^{e_i}},$$

to be the proportion of the fibre of $x \in \mathbb{Z}_{Q_{i-1}}$ in $\mathbb{Z}_{Q_i}$ that is removed at stage i.

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Now define

$$\mathbb{P}_i(x,y) := \begin{cases} \max\left\{0,\ \frac{\alpha_i(x)-\delta_i}{\alpha_i(x)(1-\delta_i)}\right\} \cdot \mathbb{P}_{i-1}(x,y), & \text{if } (x,y) \in B_i; \\ \min\left\{\frac{1}{1-\alpha_i(x)},\ \frac{1}{1-\delta_i}\right\} \cdot \mathbb{P}_{i-1}(x,y), & \text{if } (x,y) \notin B_i. \end{cases}$$

Measure removed

We use a 2nd moment calculation to bound the amount of measure removed at each stage:

Lemma

$$\mathbb{P}_{i-1}(R_{i-1}) - \mathbb{P}_i(R_i) = \mathbb{P}_i(B_i) \leq \frac{\mathbb{E}_{i-1}(\alpha_i(x)^2)}{4\delta_i(1-\delta_i)}$$

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And we estimate that 2nd moment by bounds depending on the restrictions we have on the d_i. The most general bound is:

Lemma

$$\mathbb{E}_{i-1}(\alpha_i(x)^2) \le \frac{1}{(p_i-1)^2}\prod_{j<i}\Big(1+\frac{3p_j-1}{(p_j-1)^2(1-\delta_j)}\Big).$$

The ultimate uncovered region

By tracking the measure removed at each stage we can bound

$$\mathbb{P}_k(R_k) \geq 1 - \eta := 1 - \sum_i \frac{\mathbb{E}_{i-1}(\alpha_i(x)^2)}{4\delta_i(1-\delta_i)}.$$

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However it is also possible to bound R_k in the **uniform** measure $\mathbb{P}_0$ by tracking the average logarithmic distortion $\mathbb{E}_k[\max\{\log(\mathbb{P}_k(x)/\mathbb{P}_0(x)), 0\}]$.

Lemma

$$\mathbb{P}_0(R_k) \geq (1-\eta)\exp\Big(-\frac{2}{1-\eta}\sum_{d\in D_k}\frac{1}{d}\prod_{p_i|d}\frac{1}{1-\delta_i}\Big)$$

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Many results on covering systems now reduce to choosing appropriate δ_i, and obtaining sharper bounds on the 2nd moments when necessary.

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Fibres in which the remaining measure is not sufficiently 'pseudo-random' must also be removed, otherwise problems may occur later.

Application of the method to the Hough–Nielsen 2-3 result

As an example, assume no d_j is divisible by 2 or 3. We will prove the Hough–Nielsen result that the APs cannot cover $\mathbb{Z}$ in this case. As no d_j is divisible by 2 or 3, we start with the third prime $p_3 = 5$.

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We let

$$\pi_i = \prod_{3 \le j \le i} \Big(1 + \frac{3p_j - 1}{(p_j - 1)^2(1 - \delta_j)}\Big)$$

so that $\mathbb{E}_{i-1}(\alpha_i(x)^2) \le \frac{\pi_{i-1}}{(p_i - 1)^2}$.

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We let $\mu_2 = 1$ and set

$$\mu_i = \mu_{i-1} - \frac{\pi_{i-1}}{4\delta_i(1-\delta_i)(p_i-1)^2}$$

so that, by induction, $\mathbb{P}_i(R_i) \ge \mu_i$.

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It is enough to show that $\mu_i > 0$ for all $i \ge 3$.

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Rewriting in terms of $f_i := \frac{\pi_i}{\mu_i}$ we have $f_2 = 1$ and

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We calculate the first few values assuming, say, $\delta_i = 0.2$.

k	p_k	f_k
3	5	2.32034632
4	7	4.371999733
5	11	6.569584328
	$\vdots$	
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OK, so when do we stop?

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Lemma

If $k \geq 10$, $\mu_k > 0$, *and* $f_k \leq (\log k + \log\log k - 3)^2 k$, *then the system does not cover.*

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Now note that in the 2-3 problem, this condition holds for $p_{44} = 193$:

$$f_{44} = 192.9769395 < (\log 44 + \log\log 44 - 3)^2 \cdot 44 = 196.8258827. \qquad \square$$