Critical probabilities of 1-independent percolation models

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Abstract

Given a locally finite connected infinite graph G, let the interval $[p_{\min}(G), p_{\max}(G)]$ be the smallest interval such that if $p > p_{\max}(G)$ then every 1-independent bond percolation model on G with bond probability p percolates, and for $p < p_{\min}(G)$ none does. We determine this interval for trees in terms of the branching number of the tree. We also give some general bounds for other graphs G, in particular for lattices.

1 Introduction

Let G be a locally finite connected infinite graph. A (bond) *percolation model* on G is a probability measure on the subgraphs of G. We call an edge *open* if it belongs to our random subgraph, and *closed* otherwise. In an *independent* percolation measure, the edges are open or closed independently of the states of all the other edges. A weaker condition is that of 1-independence. We say a model is 1-*independent* if for any two disjoint sets of edges S_1 and S_2 that are at distance at least 1 in G, the states of the edges in S_1 are independent of the states of the edges in S_2. (This is sometimes referred to in the literature as 1-*dependent* percolation.) We say that the model *percolates* if, with positive

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probability, there is an infinite component in our random subgraph, i.e., there is an infinite connected subgraph consisting of open edges of G.

The interest in 1-independent models stems from the fact that they naturally arise from renormalizing independent models, or more generally, models with limited range dependencies. As such, 1-independent models have become a key tool in establishing bounds on critical probabilities (see for example [2, sections 3.5 and 6.2]). Given this, it is perhaps surprising that some of the most basic questions about 1-independent models are open.

Our main interest in this paper is in the case when G is a tree. Let T be a locally finite tree and fix a root $v_0 \in V(T)$. We define the *level* $\ell(v)$ of a vertex $v \in V(T)$ to be the distance in T from v to v_0. If T is infinite, define a *flow* on T to be a non-negative function $f\colon V(T) \to \mathbb{R}$ such that for each vertex v, $f(v) = \sum_i f(v_i)$, where v_i are the children of v, i.e., $\ell(v_i) = \ell(v) + 1$ and $vv_i \in E(T)$. (One can equivalently, and perhaps more naturally, define f on the edges of T, so that $f(uv) = f(v)$ where v is a child of u.) We say that a flow f is *non-trivial* if $f(v_0) > 0$. Define the *branching number* of T by

$$\mathrm{br}(T) = \sup\{\, b : \exists \text{ a non-trivial flow } f \text{ such that } b^{\ell(v)} f(v) \text{ is bounded}\,\}$$

Note that for any infinite tree, $\mathrm{br}(T) \geq 1$, and for a regular tree of degree $k+1$, $\mathrm{br}(T) = k$. Furthermore, $\mathrm{br}(T)$ is independent of the choice of the root. The following result was proved by Lyons [5, Theorem 6.2] in 1990.

Theorem 1. *If each edge of a locally finite infinite tree T is declared to be open with probability p, independently of the states of all other edges, then if $p < 1/\mathrm{br}(T)$ there is almost surely no infinite open path from v_0, and if $p > 1/\mathrm{br}(T)$ then an infinite open path from v_0 exists with positive probability.* □

We wish to extend this result to the class of 1-independent models. Since we have no fixed model in mind, there will be a range of values of p for which some models will percolate and some do not. However, if p is sufficiently large one would expect percolation in all 1-independent models, and if p is sufficiently small, no 1-independent model should percolate. Define $\mathcal{D}_{\geq p}(G)$ to be the class of 1-independent bond percolation models on G for which each edge is open with probability at least p. Define $\mathcal{D}_{\leq p}(G)$ similarly. We write

$$\begin{aligned} p_{\max}(G) &= \sup\{p : \exists \text{ a model in } \mathcal{D}_{\geq p}(G) \text{ that does not percolate}\,\} \\ p_{\min}(G) &= \inf\{p : \exists \text{ a model in } \mathcal{D}_{\leq p}(G) \text{ that does percolate}\,\}. \end{aligned}$$

In the definitions of $p_{\max}(G)$ and $p_{\min}(G)$, it is equivalent to consider 1-independent models in which each edge probability is *exactly* p. Indeed, in any non-percolating model in $\mathcal{D}_{\geq p}(G)$, edges which occur with probability $p' > p$ can be deleted independently with probability $1 - p/p'$ resulting in a non-percolating 1-independent model whose edges are open with probability p. Similarly for percolating models in $\mathcal{D}_{\leq p}(G)$, edges can be independently added so as to ensure all edges are open with probability exactly p.

If G has a finite maximum degree, then a result of Liggett, Schonmann, and Stacey [4] shows that every model in $\mathcal{D}_{\geq p}(G)$ stochastically dominates an independent bond percolation model with probability $f(p)$, where $f(p) \to 1$ as $p \to 1$. As a consequence, if the vertices of G have finite maximum degree and the independent bond percolation model on G percolates for some $p < 1$, then $p_{\max}(G) < 1$.

Our main result is the following.

Theorem 2. *Consider a 1-independent model on a tree T in which each edge is open with probability at least p. If $\mathrm{br}(T) > 2$, suppose that $p \geq \frac{3}{4}$; if $\mathrm{br}(T) \leq 2$, suppose that $p > 1 - \frac{\mathrm{br}(T)-1}{\mathrm{br}(T)^2}$. Then with positive probability there exists an infinite open path from the root.*

We shall also show that this result is essentially best possible by proving the following.

Theorem 3. *Let T be a tree with $\mathrm{br}(T) < 2$. If $p < 1 - \frac{\mathrm{br}(T)-1}{\mathrm{br}(T)^2}$ then there exists a 1-independent model on T for which each edge is open with probability at least p, and such that T almost surely does not have an infinite open path starting at the root. For any tree T and $p < \frac{3}{4}$, there is a 1-independent model on T for which each edge is open with probability at least p, but all open components have uniformly bounded depth.*

Combining Theorems 2 and 3 we see that for any locally finite tree T

$$p_{\max}(T) = \begin{cases} 1 - \frac{\mathrm{br}(T)-1}{\mathrm{br}(T)^2}, & \mathrm{br}(T) < 2; \\ \frac{3}{4}, & \mathrm{br}(T) \geq 2. \end{cases}$$

Note that in contrast to Theorem 1, one can have 1-independent models with edge probabilities close to $\frac{3}{4}$ which still fail to percolate, even for trees with very large branching numbers.

For general graphs we prove the following weaker result.

Theorem 4. *Suppose G is a locally finite connected infinite graph. Then there is a 1-independent process on G in which each edge is open with probability at least $\frac{1}{2}$, but there is almost surely no infinite open component.*

Hence $p_{\max}(G) \geq \frac{1}{2}$ holds for any graph G. Surprisingly enough, this bound is best possible.

Theorem 5. *There exists a locally finite connected infinite graph G with $p_{\max}(G) = \frac{1}{2}$.*

Theorems 2 and 3 will be proved in Section 2, while Theorems 4 and 5 will be proved in Section 3. We give some results for $p_{\min}(G)$ for trees and general graphs in Section 4. Finally, in Section 5 we discuss the important special case when G is a lattice.

2 Determining $p_{\max}$ for trees

We start this section by showing how to construct a 1-independent model on a tree in which the probability of a path existing from the root to level N is as small as possible.

Fix p and N, and for $i = N, N-1, \dots, 0$, define c_i inductively by setting

$$c_i = \begin{cases} 1 & \text{if } i = N; \\ 1 - q/c_{i+1} & \text{if } i < N,\ c_{i+1} > q; \\ 0 & \text{if } i < N,\ c_{i+1} \le q; \end{cases} \tag{1}$$

where $q = 1 - p$. Let T be a finite tree with root v_0 and depth N. Let T_i be the set of nodes at level i, $i = 0, \dots, N$. Define the following 1-independent model on T. Assign independent 0–1 Bernoulli variables X_v to the vertices $v \in V(T)$ so that $\mathbb{P}(X_v = 1) = c_i$ when $v \in T_i$. Now declare an edge uv with $u \in T_i$, $v \in T_{i+1}$, to be closed if $X_u = 0$ and $X_v = 1$. Note that this model is clearly 1-independent, and the probability of an edge being closed is $(1 - c_i)c_{i+1} \le q$. Hence each edge is open with probability at least p. Let $\eta_v^0 = \eta_v^0(T)$ be the probability that, in this model, there is no open path in T starting from v that goes down to level N (without passing through any vertex of level less than $\ell(v)$).

Theorem 6. *Consider any 1-independent model on T in which each edge is open with probability at least p. Then the probability that there is a path in T from v down to level N is at least $1 - \eta_v^0(T)$.*

Proof. For each vertex $v \in V(T)$, let F_v be the event that a path exists from v down to level N, and let $\eta_v = \mathbb{P}(F_v^{\mathrm{c}})$ be the probability that there is no such path. Fix a vertex v and let the children of v be v_i, $i = 1, \dots, r$, and their children be v_{ij}, $j = 1, \dots, r_i$. Denote the edges between these vertices by $e_i = vv_i$ and $e_{ij} = v_iv_{ij}$. Let E_e be the event that the edge e is closed. By decomposing F_v^{c} according to the first i for which F_{v_i} holds (if any) and noting that if F_v fails but F_{v_i} holds then e_i must be closed, one obtains

$$\begin{aligned} F_v^{\mathrm{c}} \subseteq (F_{v_1} \cap E_{e_1}) \cup \big(F_{v_1}^{\mathrm{c}} \cap F_{v_2} \cap E_{e_2}\big) \cup \big(F_{v_1}^{\mathrm{c}} \cap F_{v_2}^{\mathrm{c}} \cap F_{v_3} \cap E_{e_3}\big) \cup \dots \\ \cup \big(F_{v_1}^{\mathrm{c}} \cap \dots \cap F_{v_{r-1}}^{\mathrm{c}} \cap F_{v_r} \cap E_{e_r}\big) \cup \big(F_{v_1}^{\mathrm{c}} \cap \dots \cap F_{v_r}^{\mathrm{c}}\big). \end{aligned}$$

However, $F_{v_i} \subseteq \bigcup_j F_{v_{ij}}$, and the events $F_{v_1}, \dots, F_{v_{i-1}}$, E_{e_i}, and $F_{v_{ij}}$ are all independent. Hence

$$\begin{aligned}\mathbb{P}(F^{\mathrm{c}}_{v_1} \cap \dots \cap F^{\mathrm{c}}_{v_{i-1}} \cap F_{v_i} \cap E_{e_i}) &\le \mathbb{P}(F^{\mathrm{c}}_{v_1} \cap \dots \cap F^{\mathrm{c}}_{v_{i-1}} \cap (\textstyle\bigcup_j F_{v_{ij}}) \cap E_{e_i})\\ &\le q\eta_{v_1} \dots \eta_{v_{i-1}}(1 - \textstyle\prod_j \eta_{v_{ij}}).\end{aligned}$$

Consequently we have

$$\begin{aligned}\eta_v \le q(1 - \textstyle\prod_j \eta_{1j}) &+ q\eta_{v_1}(1 - \textstyle\prod_j \eta_{2j}) + q\eta_{v_1}\eta_{v_2}(1 - \textstyle\prod_j \eta_{3j}) + \dots\\ &+ q\eta_{v_1} \dots \eta_{v_{r-1}}(1 - \textstyle\prod_j \eta_{rj}) + \eta_{v_1} \dots \eta_{v_r}. \end{aligned} \tag{2}$$

Define c_i as in (1). We claim that

$$1 - \eta_v \ge c_i(1 - \textstyle\prod_j \eta_{v_j}). \tag{3}$$

We prove this claim by reverse induction on the level i. At level N it is clear as $\eta_v = 0$. Now, assuming that the result holds at level $i+1$ and v is a vertex at level i, (2) and (3) imply that

$$\begin{aligned}c_{i+1}\eta_v &\le q(1 - \eta_{v_1}) + q\eta_{v_1}(1 - \eta_{v_2}) + q\eta_{v_1}\eta_{v_2}(1 - \eta_{v_3}) + \dots\\ &\quad + q\eta_{v_1} \dots \eta_{v_{r-1}}(1 - \eta_{v_r}) + c_{i+1}\eta_{v_1} \dots \eta_{v_r}\\ &= q + (c_{i+1} - q)\eta_{v_1} \dots \eta_{v_r}\end{aligned}$$

But then $c_{i+1}(1 - \eta_v) \ge (c_{i+1} - q)(1 - \prod \eta_{v_i})$. The claim follows since either $c_i = 0$, or $c_{i+1} > q$ and $c_i = (c_{i+1} - q)/c_{i+1}$.

For the model defined at the beginning of this section, we have equality throughout, so $1 - \eta^0_v = c_i(1 - \prod \eta^0_{v_i})$. One can check this by checking for equality at each step of the above argument, or one can obtain the result more directly as follows. At level N, $X_v = 1$, so if at level ℓ, $X_v = 0$, one definitely does not have a path to level N since on that path there would be a 0–1 transition which would result in a closed edge. On the other hand, if $\ell(v) = \ell$ and $X_v = 1$, then all edges to level $\ell + 1$ are open, and the probability that there is no path to level N is just the probability of no path from any of the children v_i of v to N. These events are independent and have probability $\eta^0_{v_i}$, so one obtains $1 - \eta^0_v = \mathbb{P}(X_v = 0)0 + \mathbb{P}(X_v = 1)(1 - \prod \eta^0_{v_i}) = c_i(1 - \prod \eta^0_{v_i})$ as required.

We now prove by reverse induction on the level that $\eta_v \le \eta^0_v$. If v is at level N then $\eta_v = \eta^0_v = 0$, and if it is at level $i < N$ then

$$1 - \eta_v \ge c_i(1 - \textstyle\prod_j \eta_{v_j}) \ge c_i(1 - \textstyle\prod_j \eta^0_{v_j}) = 1 - \eta^0_v.$$

The result follows. □

Proof of Theorem 2. By compactness it suffices to show that the probability that there is a path from level 0 to level N is bounded below by some $\varepsilon > 0$, independently of N. Fix N and consider the finite tree consisting of all vertices v of T of level at most N. Assume that $p \geq \frac{3}{4}$ and write

$$c_* = (1 + \sqrt{1-4q})/2, \tag{4}$$

where $q = 1 - p$. Note that $c_* \in [\frac{1}{2}, 1]$, $c_* > \frac{1}{4} \geq q$, and c_* is the largest solution of the equation

$$c_* = 1 - q/c_*.$$

Note also that if $\mathrm{br}(T) \leq 2$ and $p > 1 - \frac{\mathrm{br}(T)-1}{\mathrm{br}(T)^2}$ then $c_* > 1/\mathrm{br}(T)$, while if $\mathrm{br}(T) > 2$ and $p \geq \frac{3}{4}$ then $c_* \geq \frac{1}{2} > 1/\mathrm{br}(T)$. With the c_i and η_v defined as in the proof of Theorem 6, we see by induction that $c_i \geq c_*$ for all i. Hence by (3)

$$1 - \eta_v \geq c_*(1 - \textstyle\prod_j \eta_{v_j}) \tag{5}$$

holds for all v.

We now use the definition of $\mathrm{br}(T)$. Let f be a non-trivial flow on T with $b^{\ell(v)} f(v) \leq 1$ where $c_*^{-1} < b < \mathrm{br}(T)$. We show by induction on the level that $\eta_v \leq 1 - \varepsilon b^{\ell(v)} f(v)$ for some fixed $\varepsilon > 0$. At level N we require $\varepsilon b^{\ell(v)} f(v) \leq 1$, which will hold for all $\varepsilon \leq 1$. Now assuming $\ell(v) = i$ and the result holds at level $i+1$, (5) gives

$$\begin{aligned}
1 - \eta_v &\geq c_*(1 - \textstyle\prod_j (1 - \varepsilon b^{i+1} f(v_j))) \\
&\geq c_*(1 - \exp(-\textstyle\sum_j \varepsilon b^{i+1} f(v_j))) \\
&= c_*(1 - \exp(-b\varepsilon b^i f(v))) \\
&\geq c_* b\varepsilon b^i f(v)/(1 + b\varepsilon b^i f(v)) \\
&\geq c_* b\varepsilon b^i f(v)/(1 + b\varepsilon),
\end{aligned}$$

where we have used $1/(1+x) \geq e^{-x} \geq 1 - x$ for $x \geq 0$, and $b^i f(v) \leq 1$. Now if we choose ε sufficiently small so that $c_* b \geq 1 + b\varepsilon$, we have $1 - \eta_v \geq \varepsilon b^i f(v)$, so $\eta_v \leq 1 - \varepsilon b^{\ell(v)} f(v)$. Finally, for $v = v_0$, we have $\eta_{v_0} \leq 1 - \varepsilon f(v_0)$, which is bounded away from 1, independently of N. Hence $1 - \eta_{v_0} \geq \varepsilon f(v_0)$ is bounded away from zero, as required. □

Proof of Theorem 3. Assume first that $\mathrm{br}(T) < 2$ and $\frac{3}{4} \leq p < 1 - \frac{\mathrm{br}(T)-1}{\mathrm{br}(T)^2}$. Define c_* as in (4), so that $p = 1 - c_*(1 - c_*)$ and note that $c_* < 1/\mathrm{br}(T)$. Construct a model by assigning independent 0–1 Bernoulli variables X_v to each vertex v which are 1 with probability c_*. An edge uv is closed if $X_u = 0$ and $X_v = 1$, where $\ell(v) = \ell(u) + 1$. Note that each edge is closed with probability $c_*(1 - c_*) = 1 - p$, and the model is 1-independent. Suppose that an infinite open component exists. Then there is an infinite path $v_0 v_1 \ldots$ such that the sequence $X_{v_0}, X_{v_1}, \ldots$ never contains a 1 followed by a 0. But then the X_{v_i} must be eventually constant, and so the site percolation model determined by the X_v must have an infinite component of 1s, or an infinite component of 0s. Neither is possible since $1 - c_* \leq c_* < 1/\mathrm{br}(T)$. (The critical probability for independent site percolation on a tree is the same as for independent bond percolation, which is $1/\mathrm{br}(T)$ by Theorem 1.)

Now assume $p < \frac{3}{4}$. If N is large enough, then the sequence c_i defined in (1) is zero at $i = 0$. Indeed, by the arithmetic-geometric mean inequality $2\sqrt{q} \leq q/c + c$, so $1 - q/c \leq c - (2\sqrt{q} - 1)$. Hence for $q > \frac{1}{4}$, c_i decreases at each step by at least $2\sqrt{q} - 1 > 0$ until it becomes zero. Now on the infinite tree, define c_i as $c_{i \bmod (N+1)}$, and assign 0–1 Bernoulli variables X_v to each vertex as in Section 2 so that at level i, $\mathbb{P}(X_v = 1) = c_i$. Once again, declare an edge uv closed if $X_u = 0$ and $X_v = 1$, where $\ell(v) = \ell(u) + 1$. The probability that an edge is closed is at most q when $\ell(u) \not\equiv N \bmod (N+1)$, and zero when $\ell(u) \equiv N \bmod (N+1)$. Also, there is no open path from any vertex at level $k(N+1)$ to level $k(N+1) + N$. Hence any open component is of uniformly bounded depth. □

3 Bounds on $p_{\max}$ for arbitrary graphs

Proof of Theorem 4. Fix a vertex v_0 of G and a (deterministic) vertex labelling $c\colon V(G) \to [0,1]$ defined by

$$c(v) = \begin{cases} 0 & \text{if } d(v, v_0) \equiv 0 \bmod 4, \\ 1 & \text{if } d(v, v_0) \equiv 2 \bmod 4, \\ \frac{1}{2} & \text{if } d(v, v_0) \equiv 1, 3 \bmod 4, \end{cases}$$

where $d(v, v_0)$ is the graph distance from v to v_0. Now define independent 0–1 Bernoulli random variables X_v for each $v \in V(G)$ so that

$$\mathbb{P}(X_v = 1) = c(v).$$

Declare a bond uv of G to be open if $X_u = X_v$. Then the process on bonds is 1-independent and the probability of an edge being open is at least $\frac{1}{2}$. Indeed, if $c(u) = \frac{1}{2}$ then $\mathbb{P}(X_u = s) = \frac{1}{2}$ for either $s \in \{0, 1\}$ so the bond is open with probability $\frac{1}{2}$. Similarly if $c(v) = \frac{1}{2}$. If $c(u), c(v) \in \{0, 1\}$ then $c(u) = c(v)$ since no vertex with $c(w) = 0$ is adjacent to any vertex with $c(w) = 1$. Then uv is open with probability 1.

Any open cluster in G must consist of sites with the same value of X_w. Thus the distances from v_0 of the vertices of this cluster cannot cross between

$4k+1$ and $4k+3$ if $X_w = 0$, or between $4k+3$ and $4k+5$ if $X_w = 1$. Thus the points of the cluster have bounded distance from v_0. Thus all open clusters are finite. $\square$

To prove Theorem 5 we shall use the following.

Lemma 7. *Let $\varepsilon > 0$. Then for sufficiently large n the following holds. Given any 1-independent model on the complete bipartite graph $K_{n,n}$ in which each edge is open with probability at least $\frac{1}{2} + \varepsilon$, then with probability at least $1 - \varepsilon$ there exists an open component containing at least a fraction $\frac{1}{2} + \frac{\varepsilon}{2}$ of both bipartite classes.*

Proof. Decompose the edge set of $G = K_{n,n}$ as the union of n perfect matchings $M_1, \ldots, M_n$ and let m_i be the number of open edges in M_i. Then as the edges in M_i are independent, m_i stochastically dominates a binomial random variable with parameters n and $\frac{1}{2} + \varepsilon$. Thus by Hoeffding's inequality

$$\mathbb{P}\big(m_i < (\tfrac{1}{2} + \tfrac{\varepsilon}{2})n\big) < \exp(-\varepsilon^2 n/2).$$

Thus if m is the total number of open edges in G,

$$\mathbb{P}\big(m < (\tfrac{1}{2} + \tfrac{\varepsilon}{2})n^2\big) \le \mathbb{P}\big(\exists i\colon m_i < (\tfrac{1}{2} + \tfrac{\varepsilon}{2})n\big) \le n \exp(-\varepsilon^2 n/2),$$

which is at most ε when n is sufficiently large.

Now suppose that $m \ge (\frac{1}{2} + \frac{\varepsilon}{2})n^2$. Let the bipartite classes of G be A and B and suppose the open components are $C_i = G[A_i \cup B_i]$, $i = 1, \ldots, c$, where $\{A_i : i = 1, \ldots, c,\ A_i \ne \emptyset\}$ and $\{B_i : i = 1, \ldots, c,\ B_i \ne \emptyset\}$ are partitions of A and B respectively. Let $a_i = |A_i|$ and $b_i = |B_i|$. Then $m \le \sum a_i b_i$.

Suppose first that $a_i < (\frac{1}{2}+\frac{\varepsilon}{2})n$ for every i. Then $m < (\frac{1}{2}+\frac{\varepsilon}{2})n \sum b_i = (\frac{1}{2}+\frac{\varepsilon}{2})n^2$, a contradiction. Thus, without loss of generality, we may assume that $a_1 \ge (\frac{1}{2}+\frac{\varepsilon}{2})n$. Similarly we may assume that $b_j \ge (\frac{1}{2}+\frac{\varepsilon}{2})n$ for some j. If $j = 1$ we are done, so without loss of generality assume $j = 2$. As $a_i \le n - a_1 < a_1$ for all $i > 1$, $\sum_{i \ne 2} a_i b_i \le a_1(n - b_2)$, while $a_2 b_2 \le (n - a_1)b_2$. Hence

$$m \le a_1(n - b_2) + (n - a_1)b_2 = \tfrac{n^2}{2} - 2(a_1 - \tfrac{n}{2})(b_2 - \tfrac{n}{2}) < \tfrac{n^2}{2},$$

a contradiction. Hence there exists an open component meeting at least a fraction $\frac{1}{2} + \frac{\varepsilon}{2}$ of both bipartite classes. $\square$

Proof of Theorem 5. By Theorem 4 it is enough to give an example of a graph G such that for any $p > \frac{1}{2}$, every model in $\mathcal{D}_{\ge p}(G)$ percolates.

Let T be the infinite binary tree, and let G be obtained by replacing each vertex v of T by $\ell(v)$ copies $v_1, \ldots, v_{\ell(v)}$, and each edge uv by a complete bipartite graph consisting of all edges $u_i v_j$, $1 \le i \le \ell(u)$, $1 \le j \le \ell(v)$.

Consider a model in $\mathcal{D}_{\geq p}(G)$, where $p = 1/2 + \epsilon > 1/2$. We proceed by renormalizing this model to give a model on T. Specifically, for each edge uv in T, declare uv to be open if there exists an open component in the complete bipartite graph $G[\{u_1, \dots, u_{\ell(u)}, v_1, \dots, v_{\ell(v)}\}]$ which contains more than $\ell(u)/2$ of the vertices $u_1, \dots, u_{\ell(u)}$ and more than $\ell(v)/2$ of the vertices $v_1, \dots, v_{\ell(v)}$. This clearly gives a 1-independent model on T. Moreover, the existence of an infinite open path in T implies the existence of an infinite open component in G.

Now assume u and v are at levels n and $n+1$, where n is sufficiently large. Then the graph $G[\{u_1, \dots, u_{\ell(u)}, v_1, \dots, v_{\ell(v)}\}]$ is isomorphic to $K_{n,n+1}$. Ignoring one of the vertices in the larger class, Lemma 7 implies that this subgraph will have an open component meeting more than $(n+1)/2$ vertices of each bipartite class with probability at least $1 - \varepsilon$. Thus for $\varepsilon < \frac{1}{4}$, uv will be open with probability more than $\frac{3}{4}$. Theorem 2 then implies that there is percolation in (a sufficiently deep subtree of) T and hence there is percolation in G. □

One might imagine that choosing a tree with higher branching number might help in the proof of Theorem 5, but in fact any tree T with $\mathrm{br}(T) > 1$ will work.

4 Bounds on $p_{\min}$.

First we prove an upper bound on $p_{\min}(G)$ that applies to an arbitrary locally finite graph G.

Proposition 8. *If G is a locally finite connected infinite graph then $p_{\min}(G) \leq p_{\text{site}}(G)^2$ where $p_{\text{site}}(G)$ is the critical probability for independent site percolation on G.*

Proof. Consider the model which declares each site open independently with probability $\sqrt{p}$, and then declares each bond open if it joins two open sites. Each bond is open with probability p, and the bonds are 1-independent. The bonds form infinite open clusters precisely when the sites do, so this model percolates for $p > p_{\text{site}}(G)^2$. □

For trees we show that the above bound is in fact sharp.

Theorem 9. *For any locally finite tree T, $p_{\min}(T) = 1/\mathrm{br}(T)^2$.*

Proof. By Proposition 8, $p_{\min}(T) \leq p_{\text{site}}(T)^2$. As site percolation is equivalent to bond percolation on trees, Theorem 1 implies $p_{\min}(T) \leq 1/\mathrm{br}(T)^2$.

For the converse, consider a 1-independent model with edge probability at most p. Assume $v \in V(T)$ has children v_i, and their children are v_{ij}. If we let ζ_v be the probability that an infinite open path exists from v downwards, then we may assume for contradiction that ζ_v is non-zero when $v = v_0$. Also, if an infinite path exists from v then at least one of the edges vv_i must be open and at least one of the v_{ij} must have an infinite open path from it. Since the openness of vv_i is independent of the existence of an open path from v_{ij}, we have

$$\zeta_v \le \textstyle\sum_{i,j} p \zeta_{v_{ij}}.$$

Now define a flow $f\colon V(T) \to \mathbb{R}$ on T. We set $f(v_0) = \zeta_{v_0}$, and inductively define f on vertices at even levels by

$$f(v_{ij}) = \frac{\zeta_{v_{ij}}}{\sum_{kl} \zeta_{v_{kl}}} f(v).$$

(If $\sum_{kl} \zeta_{v_{kl}} = 0$ then $\zeta_v = 0$, so $f(v) = 0$, and we take $f(v_{ij}) = 0$.) To complete the definition of f, we define f at odd levels by

$$f(v_i) = \textstyle\sum_j f(v_{ij}).$$

It is clear that f is a flow on T. We also note that at even levels

$$f(v_{ij}) = \frac{\zeta_{v_{ij}}}{\sum_{kl} \zeta_{v_{kl}}} f(v) \le \frac{\zeta_{v_{ij}}}{\zeta_v} p f(v),$$

so by induction $f(v) \le \zeta_v p^{\ell(v)/2} \le p^{\ell(v)/2}$. For odd levels $f(v_i) \le f(v) \le p^{(\ell(v_i)-1)/2}$. Thus if $\zeta_{v_0} > 0$ then $p^{-1/2} \le \mathrm{br}(T)$ and so $p \ge 1/\mathrm{br}(T)^2$. As this holds for any 1-independent model that percolates, $p_{\min}(T) \ge 1/\mathrm{br}(T)^2$. □

We finish this section by noting that the inequality in Proposition 8 may be strict. Indeed, this is clear as $p_{\min}(G) \le p_{\mathrm{bond}}(G)$, where $p_{\mathrm{bond}}(G)$ is the critical probability for independent bond percolation, and there are examples of graphs G for which $p_{\mathrm{bond}}(G) = 0$ but $p_{\mathrm{site}}(G) = 1$. We now present an even more dramatic example.

Theorem 10. *There exists a locally finite connected infinite graph G with $p_{\min}(G) = 0$, but $p_{\mathrm{bond}}(G) = p_{\mathrm{site}}(G) = 1$.*

Proof. Define G to be a bipartite graph with one vertex class $\{v_1, v_2, \dots\}$ and the other vertex class a union of sets of vertices $U_1, U_2, \dots$. Join every vertex in U_k to both v_k and v_{k+1} (see Figure 1). Assume $|U_k| = q_k^2 + q_k + 1$, where q_k is a prime-power; in a moment we shall consider each U_k as the set of vertices of a projective plane. We shall assume $q_k \to \infty$ sufficiently slowly so that $|U_k| = o(\log k)$.

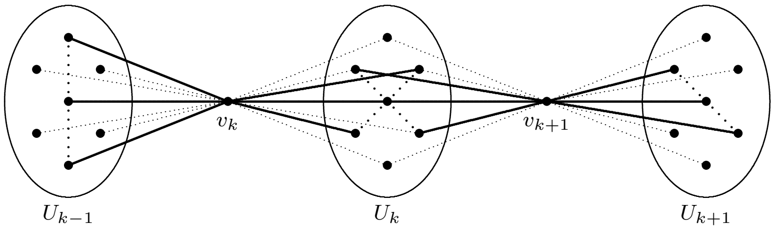


Figure 1: Graph G with $p_{\min}(G) = 0$ but $p_{\text{bond}}(G) = p_{\text{site}}(G) = 1$.

It is clear that $p_{\text{site}}(G) = 1$. Indeed $p_{\text{bond}}(G) = 1$, since if each edge is open independently with probability $p < 1$ then the probability of an infinite open component containing v_k is $\prod_{i \geq k}(1 - (1-p^2)^{|U_i|}) = \prod_{i \geq k}(1 - e^{\lambda |U_i|})$ for some $\lambda > 0$. However, as $|U_k| = o(\log k)$, $e^{\lambda |U_i|} = \Omega(1/i)$, so this product converges to zero for any $p < 1$.

We now show that $p_{\min}(G) = 0$. Fix $p > 0$. If $(q_k + 1)/(q_k^2 + q_k + 1) > p$ declare all edges incident to U_k closed. If $(q_k + 1)/(q_k^2 + q_k + 1) \leq p$, declare open all edges from v_k to a projective line in U_k chosen uniformly at random from the set of projective lines in U_k. Similarly declare open all edges from v_{k+1} to an independently chosen projective line in U_k. Note that this model is 1-independent and each edge is open with probability at most p. As any two lines in U_k intersect, there will be an open path from v_k to v_{k+1} for all sufficiently large k, and hence there will always be an infinite open component. Since $p > 0$ was arbitrary, $p_{\min}(G) = 0$. □

5 1-independent percolation on lattices

In this section we discuss 1-independent percolation on lattices. Let $\mathbb{Z}^d$ denote the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and edges joining pairs of vertices that are (Euclidean) distance 1 apart. It is easy to see that $p_{\max}(\mathbb{Z}^d) < 1$, but giving a good upper bound for $p_{\max}(\mathbb{Z}^d)$ is surprisingly difficult. In [1, Theorem 2] the following was proved.

Theorem 11. *For the lattice* $\mathbb{Z}^2$, $p_{\max}(\mathbb{Z}^2) \leq 0.8639$. □

We now give an example found by Chuck Newman (see [6]) of a 1-independent model on $\mathbb{Z}^2$ which shows that

$$p_{\max}(\mathbb{Z}^2) \geq p_{\text{site}}(\mathbb{Z}^2)^2 + (1 - p_{\text{site}}(\mathbb{Z}^2))^2 > \tfrac{1}{2}.$$

Consider an independent site percolation with sites open with probability ρ. Declare a bond to be open if it joins two sites in the same state (either both open or both closed). Then each bond is open with probability $p = \rho^2 + (1-\rho)^2$. An infinite open cluster would give either an infinite cluster of open sites or an infinite cluster of closed sites in the site percolation model. Thus if $p_{\mathrm{site}}(\mathbb{Z}^2) > \rho > 0.5$ the 1-independent model will not percolate. Thus we have a model that does not percolate for p below $p_{\mathrm{site}}(\mathbb{Z}^2)^2+(1-p_{\mathrm{site}}(\mathbb{Z}^2))^2$.

Since $0.556 \le p_{\mathrm{site}} \le 0.679492$ [3, 8], we obtain

$$0.5062 \le p_{\max}(\mathbb{Z}^2) \le 0.8639.$$

Using the (non-rigourous) estimate $p_{\mathrm{site}} \approx 0.592746$ [9, 1], the lower bound can be improved to $p_{\max}(\mathbb{Z}^2) \ge 0.5172$. As the upper and lower bounds for $p_{\max}(\mathbb{Z}^2)$ are still far apart, we pose the following question.

Question 1. *What is the value of* $p_{\max}(\mathbb{Z}^2)$*?*

For $\mathbb{Z}^d$ we note that $p_{\max}(\mathbb{Z}^d)$ is a decreasing function of d since absence of percolation in $\mathbb{Z}^d$ implies absence of percolation in any $\mathbb{Z}^{d-1}$ subspace. Thus $p_{\max}(\mathbb{Z}^d)$ tends to limit as $d \to \infty$, which is at least $\frac{1}{2}$ by Theorem 4. This suggests another question.

Question 2. *What is the limit of* $p_{\max}(\mathbb{Z}^d)$ *as* $d \to \infty$*?*

We now consider $p_{\min}(G)$. It is easy to prove a lower bound for every lattice in terms of the *connective constant* μ, which is defined by the requirement that the number c_n of self-avoiding walks of length n starting from a given vertex is given by $c_n = (\mu + o(1))^n$.

Proposition 12. *For any locally finite connected infinite graph G for which the connective constant μ exists, $p_{\min}(G) \ge 1/\mu^2$.*

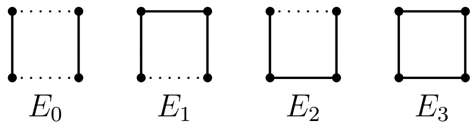
Proof. If there is an infinite open cluster then, with positive probability, there must be an infinite open cluster containing a given vertex O. Thus there must be an infinite induced path starting at O in the subgraph consisting of open edges. Assume $p < 1/\mu^2$, where μ is the connective constant. Fix any self-avoiding walk P of the lattice of edge-length $2n$. By taking every other edge of P, we get a set of independent edges of size n. Thus the probability that P is open is at most p^n. But if c_{2n} is the number of such walks then $c_{2n} = (\mu + o(1))^{2n}$. Thus the expected number of open self-avoiding walks is at most $(p\mu^2 + o(1))^n$. Since $p < 1/\mu^2$, this tends to 0. So the probability of an infinite open path starting at O is zero. □

We note that Proposition 12 applies in much more generality than just for the graphs $\mathbb{Z}^d$. For example, it suffices to assume that the graph G has a vertex transitive automorphism group.

For $\mathbb{Z}^2$, Pönitz and Tittman [7] proved that $\mu \le 2.679192495$, which gives the bound $p_{\min}(\mathbb{Z}^2) \ge 0.1393$. Proposition 8 shows that $p_{\min}(\mathbb{Z}^2) \le p_{\mathrm{site}}(\mathbb{Z}^2)$. Using the known bounds on $p_{\mathrm{site}}(\mathbb{Z}^2)$ we obtain $p_{\min}(\mathbb{Z}^2) \le 0.3514$ (non-rigorously) or $p_{\min}(\mathbb{Z}^2) \le 0.4618$ (rigorously).

For large d, $\mu(\mathbb{Z}^d)^{-1} \sim p_{\mathrm{site}}(\mathbb{Z}^d) \sim \frac{1}{2d}$, so Propositions 8 and 12 give $p_{\min}(\mathbb{Z}^d) \sim \frac{1}{4d^2}$ as $d \to \infty$. In this case the upper and lower bounds are fairly close.

We do not believe that the lower bound $1/\mu^2$ is best possible. To give a heuristic argument, consider the lattice $\mathbb{Z}^2$ and assume the 1-independent model is invariant under translation and rotation by 90°. Each edge now has the same probability p of being open. Consider the probabilities of the following four events (where dotted lines indicate closed edges and solid lines indicate open edges).



Clearly $\sum \mathbb{P}(E_i) = p^2$ since $\bigcup E_i$ is the event that two independent vertical edges are open. However, $\mathbb{P}(E_1) = \mathbb{P}(E_2)$ by symmetry, so $\mathbb{P}(E_1) = \mathbb{P}(E_2) \le p^2/2$. Following the proof of Proposition 12, consider the event that a self-avoiding walk $P = (e_1, \dots, e_{2n})$ is an induced path in the subgraph of open edges. Inductively remove edges e_{2k} from P unless the edges $e_{2k-1}, e_{2k}, e_{2k+1}$ form 3 edges of a unit square. In this case remove e_{2k+2} and continue with edge e_{2k+4}. In this way we decompose a subgraph of P into $n-2r$ independent edges and r paths of length 3. If P is induced, then the fourth edges must be closed in all the squares made from the paths of length 3. The probability that P is open and induced is therefore at most $p^{n-2r}(p^2/2)^r = p^n/2^r$. It is easy to show that there is some $\varepsilon > 0$ such that there are at most $(\mu - \varepsilon + o(1))^{2n-1}$ self-avoiding walks P with $r < \varepsilon n$. Thus the expected number of induced open paths P is at most

$$p^n(\mu - \varepsilon + o(1))^{2n-1} + (p/2^{\varepsilon})^n(\mu + o(1))^{2n-1} \le (p\mu^2 - \varepsilon' + o(1))^n$$

for some $\varepsilon' > 0$. Thus for percolation we would need $p \ge (1+\varepsilon')/\mu^2$.

Needless to say, questions can be asked about $p_{\min}(G)$ and $p_{\max}(G)$ for many other graphs G. It is worth noting that all the examples given in this paper are not just 1-independent, but are *two-block factor* models as defined by Liggett, Schonmann, and Stacey [4]. It would be interesting to know if there are examples of graphs for which $p_{\min}(G)$ or $p_{\max}(G)$ change if we restrict the set of models considered to just two-block factor models.

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