Covering systems

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Julian Sahasrabudhe, and Marius Tiba.

Covering Systems

A **covering system** is a finite collection of arithmetic progressions

$$(a_i \bmod d_i) := a_i + d_i\mathbb{Z}, \qquad i = 1, \dots, k,$$

that cover $\mathbb{Z}$:

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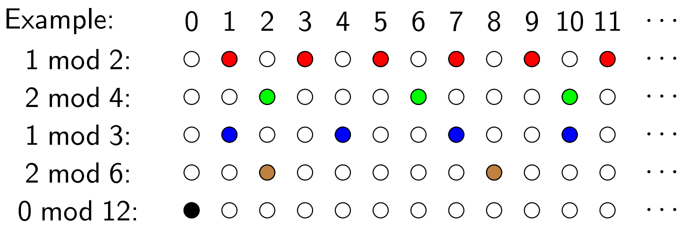
We are interested in the case when the moduli d_i are **distinct**, say

$$1 < d_1 < d_2 < \cdots < d_k.$$

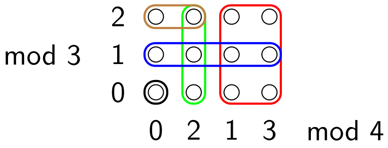
Covering Systems

Example:	0	1	2	3	4	5	6	7	8	9	10	11	···
1 mod 2:	○	●	○	●	○	●	○	●	○	●	○	●	···
2 mod 4:	○	○	●	○	○	○	●	○	○	○	●	○	···
1 mod 3:	○	●	○	○	●	○	○	●	○	○	●	○	···
2 mod 6:	○	○	●	○	○	○	○	○	●	○	○	○	···
0 mod 12:	●	○	○	○	○	○	○	○	○	○	○	○	···

Covering Systems



Or using the CRT: $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$



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If all the moduli d_i are distinct, can the smallest one $d_1 = \min d_i$ be arbitrarily large?

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If all the moduli d_i are distinct, $d_1 < 10^{16}$.

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We give a much simpler proof of this result, improving it to:

Theorem (BBMST, 2018)

If all the moduli d_i are distinct, $d_1 < 616000$.

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Our techniques give a simple proof of this, and strengthens it to:

Theorem (BBMST, 2018)

In any covering system, $Q = \operatorname{lcm}\{d_i\}$ is divisible by 2, 9, or 15.

Odd moduli

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Any covering system with distinct **square free** *moduli must contain an even modulus.*

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This last result is much harder to prove with our methods.

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In any covering system with there must exists $i \neq j$ with $d_i \mid d_j$.

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Is it true that for any $C > 0$, if $d_i \geq n_0$ and $\sum \frac{1}{d_i} < C$ then the density of the uncovered set is $> f(C) > 0$ for all sufficiently large n_0?

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Clearly the uncovered set can be made to have density $\leq \prod(1 - \frac{1}{d_i})$ by choosing the a_i greedily, so $\sum \frac{1}{d_i} < C$ is a necessary condition.

Extensions of Erdős–Graham

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Unfortunately it is not sufficient:

Theorem (BBMST, 2018)

For any $\varepsilon > 0$, $n_0 > 0$, there exists a system of APs with distinct moduli d_i, $\sum \frac{1}{d_i} < 1$ and $d_i \geq n_0$, with the density of the uncovered set $< \varepsilon$.

Proof: explicit construction.

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Theorem (BBMST, 2018)

If $\mu(d)$ is the multiplicative function defined by $\mu(p^e) = 1 + \frac{1}{p}(\log p)^{3+\varepsilon}$, then there exists $n_0 > 0$ such that for any C and any system of APs with distinct moduli d_i with $\sum \frac{\mu(d_i)}{d_i} < C$, $d_i \geq n_0$, the density of the uncovered set is $> e^{-4C}/2$.

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This immediately implies Hough's result, that the minimum modulus in a covering system is bounded.

It also implies the original Erdős–Graham conjecture.

Extensions of Erdős–Graham

We also show that this result is close to best possible.

Theorem (BBMST, 2018)

For any $\lambda > 0$, if $\mu(d)$ is the multiplicative function defined by $\mu(p^e) = 1 + \frac{\lambda}{p}$, then for any $n_0 > 0$, $\varepsilon > 0$, $C > 0$, there exists a system of APs with distinct moduli such that $\sum \frac{\mu(d_i)}{d_i} < C$, $d_i \geq n_0$, and the density of the uncovered set is $< \varepsilon$.

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This leaves the exact threshold of $\mu(p^e)$ open. We know the threshold must lie between

$$1 + \frac{\Omega(1)}{p} \qquad \text{and} \qquad 1 + \frac{(\log p)^{3+o(1)}}{p}.$$

Setup for the proofs

Write $Q = \mathrm{lcm}\{d_i\} = p_1^{e_1} \dots p_n^{e_n}$. We can think of a covering system as a cover of the hypercuboid

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We identify subsets

$$S \subseteq \mathbb{Z}_{Q_i} = \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_i^{e_i}}$$

with the subset

$$S \times \mathbb{Z}_{p_{i+1}^{e_{i+1}}} \subseteq \mathbb{Z}_{Q_{i+1}}$$

or

$$S \times \mathbb{Z}_{p_{i+1}^{e_{i+1}}} \times \dots \mathbb{Z}_{p_i^{e_i}} \subseteq \mathbb{Z}_Q$$

and we identify arithmetic progressions $(a_j \bmod d_j)$, $d_j \mid Q_i$, with the corresponding subset of $\mathbb{Z}_{Q_i}$.

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$R_i = \mathbb{Z}_Q \setminus \bigcup_{d_j \in D_i} (a_j \bmod d_j) = R_{i-1} \setminus B_i$
be the subset of $\mathbb{Z}_Q$ remaining after stage i.

Setup for the proofs

We construct a sequence of probability measures $\mathbb{P}_i$ on $\mathbb{Z}_Q = \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_n^{e_n}}$ which is uniform on each fibre of $x \in \mathbb{Z}_{Q_i}$, i.e., it is a product of a (non-trivial) measure on $\mathbb{Z}_{Q_i}$ (which we also call $\mathbb{P}_i$) with the uniform measure on $\mathbb{Z}_{Q/Q_i}$.

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$\mathbb{P}_i(x, y)$ is as small as possible on the removed set B_i.

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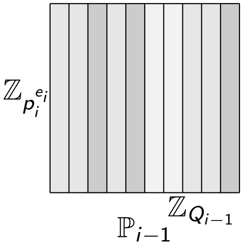

$\mathbb{P}_{i-1}$ is defined on $\mathbb{Z}_{Q_{i-1}}$ and extended uniformly on each fibre $\{x\} \times \mathbb{Z}_{p_i^{e_i}} \subseteq \mathbb{Z}_{Q_{i-1}} \times \mathbb{Z}_{p_i^{e_i}} = \mathbb{Z}_{Q_i}$.

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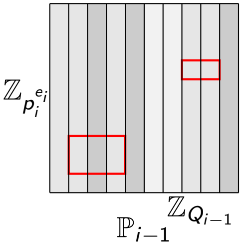

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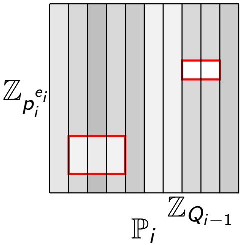

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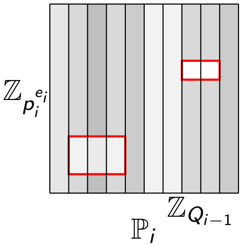

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$\mathbb{P}_i$ is defined to give the same mass to each fibre, but shifted away from B_i.

But not so much that it increases the density by more than $1/(1-\delta_i)$ anywhere. Note that if less than δ_i of the fibre is removed, then no measure is placed inside B_i in that fibre.

Setup for the proofs

Formally, for each $x \in \mathbb{Z}_{Q_{i-1}}$, define

$$\alpha_i(x) = \frac{\mathbb{P}_{i-1}\big(x \cap B_i\big)}{\mathbb{P}_{i-1}(x)} = \frac{\big|\big\{y \in \mathbb{Z}_{p_i^{e_i}} : (x, y) \in B_i\big\}\big|}{p_i^{e_i}},$$

to be the proportion of the fibre of $x \in \mathbb{Z}_{Q_{i-1}}$ in $\mathbb{Z}_{Q_i}$ that is removed at stage i.

Setup for the proofs

Formally, for each $x \in \mathbb{Z}_{Q_{i-1}}$, define

$$\alpha_i(x) = \frac{\mathbb{P}_{i-1}\big(x \cap B_i\big)}{\mathbb{P}_{i-1}(x)} = \frac{\Big|\Big\{y \in \mathbb{Z}_{p_i^{e_i}} : (x,y) \in B_i\Big\}\Big|}{p_i^{e_i}},$$

to be the proportion of the fibre of $x \in \mathbb{Z}_{Q_{i-1}}$ in $\mathbb{Z}_{Q_i}$ that is removed at stage i.

Now define

$$\mathbb{P}_i(x,y) := \begin{cases} \max\Big\{0,\ \frac{\alpha_i(x)-\delta_i}{\alpha_i(x)(1-\delta_i)}\Big\} \cdot \mathbb{P}_{i-1}(x,y), & \text{if } (x,y) \in B_i; \\ \min\Big\{\frac{1}{1-\alpha_i(x)},\ \frac{1}{1-\delta_i}\Big\} \cdot \mathbb{P}_{i-1}(x,y), & \text{if } (x,y) \notin B_i. \end{cases}$$

Measure removed

We use a 2nd moment calculation to bound the amount of measure removed at each stage:

Lemma

$$\mathbb{P}_{i-1}(R_{i-1}) - \mathbb{P}_i(R_i) = \mathbb{P}_i(B_i) \leq \frac{\mathbb{E}_{i-1}(\alpha_i(x)^2)}{4\delta_i(1-\delta_i)}$$

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Proof: $\mathbb{P}_i(R_{i-1}) = \mathbb{P}_{i-1}(R_{i-1})$ as R_{i-1} is a union of complete fibres. Thus the measure removed is just $\mathbb{P}_i(R_{i-1} \setminus R_i) = \mathbb{P}_i(B_i)$, the $\mathbb{P}_i$-measure remaining inside B_i. But this is exactly

$$\mathbb{P}_i(B_i) = \mathbb{E}_{i-1}\frac{\max\{\alpha_i(x) - \delta_i, 0\}}{1-\delta_i}$$

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$$\mathbb{P}_i(B_i) = \mathbb{E}_{i-1} \frac{\max\{\alpha_i(x) - \delta_i, 0\}}{1-\delta_i}$$

and we can bound $\alpha - \delta \le \alpha^2/4\delta$ for all $\alpha > 0$ by rearranging the inequality $(\alpha - 2\delta)^2 \ge 0$.

Bounding the second moment

Write
$$\nu(d) = \prod_{p_j \mid d} \frac{1}{1-\delta_j}$$

Lemma

$$\mathbb{P}_i(a \bmod d) \le \frac{1}{d} \prod_{j \le i,\, p_j \mid d} \frac{1}{1-\delta_j} = \frac{\nu(\gcd(Q_i, d))}{d}.$$

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Lemma

$$\mathbb{E}_{i-1}(\alpha_i(x)^2) \le \sum_{m_1 p_i^j,\, m_2 p_i^k \in N_i} p_i^{-j-k} \frac{\nu(\mathrm{lcm}(m_1, m_2))}{\mathrm{lcm}(m_1, m_2)}$$
$$\le \frac{1}{(p_i - 1)^2} \prod_{j<i} \Big(1 + \frac{3p_j - 1}{(p_j - 1)^2 (1-\delta_j)}\Big).$$

The ultimate uncovered region

By tracking the measure removed at each stage we can bound

$$\mathbb{P}_k(R_k) \geq 1 - \eta := 1 - \sum_i \frac{\mathbb{E}_{i-1}(\alpha_i(x)^2)}{4\delta_i(1-\delta_i)}.$$

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If this is positive we know that there is an uncovered region.

However it is also possible to bound R_k in the **uniform** measure $\mathbb{P}_0$ by tracking the average logarithmic distortion $\mathbb{E}_k[\max\{\log(\mathbb{P}_k(x)/\mathbb{P}_0(x)), 0\}]$.

Lemma

$$\mathbb{P}_0(R_k) \geq (1-\eta)\exp\Big(-\frac{2}{1-\eta}\sum_{d\in D_k}\frac{\nu(d)}{d}\Big)$$

Differences from Hough's method

In Hough's method, probability measures $\mathbb{P}_i$ are also defined. However measure is removed entirely from B_i, and then renormalized on the rest of the fibre.

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At each stage, the Lovász Local Lemma is needed to estimate the amount of measure in B_i, as B_i is now a much more complicated set.

Fibres in which the remaining measure is not sufficiently 'pseudo-random' must also be removed, otherwise problems may occur later.

Application of the method to the Hough–Nielsen 2-3 result

As an example, assume no d_j is divisible by 2 or 3. We will prove the Hough–Nielsen result that the APs cannot cover $\mathbb{Z}$ in this case. As no d_j is divisible by 2 or 3, we start with the third prime $p_3 = 5$.

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We let

$$\pi_i = \prod_{3 \le j \le i} \Big(1 + \frac{3p_j - 1}{(p_j - 1)^2(1 - \delta_j)}\Big)$$

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We let $\mu_2 = 1$ and set
$$\mu_i = \mu_{i-1} - \frac{\pi_{i-1}}{4\delta_i(1-\delta_i)(p_i-1)^2}$$
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It is enough to show that $\mu_i > 0$ for all $i \ge 3$.

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Rewriting in terms of $f_i := \frac{\pi_i}{\mu_i}$ we have $f_2 = 1$ and

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We calculate the first few values assuming, say, $\delta_i = 0.2$.

k	p_k	f_k
3	5	2.32034632
4	7	4.371999733
5	11	6.569584328
	$\vdots$	
44	193	192.9769395
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OK, so when do we stop?

Application of the method to the Hough–Nielsen 2-3 result

Lemma

If $k \geq 10$, $\mu_k > 0$, and $f_k \geq (\log k + \log\log k - 3)^2 k$, then the system does not cover.

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Proof: we use the result that $p_k \geq (\log k + \log\log k - 1)k$ (Dusart, 1999) to prove by induction that this assumption on μ_k, f_k implies the same for $k + 1$.

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Now note that in the 2-3 problem, this condition holds for $p_{44} = 193$:

$$f_{44} = 192.9769395 < (\log 44 + \log\log 44 - 3)^2 \cdot 44 = 196.8258827. \qquad \square$$