Pair dominating graphs

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Abstract

An oriented graph *dominates pairs* if for every pair of vertices u, v, there exists a vertex w such that the edges $\vec{wu}$ and $\vec{wv}$ both lie in G. We construct regular oriented triangle free graphs with this property, and thereby we disprove a conjecture of Myers. We also construct oriented graphs for which each pair of vertices is dominated by a unique vertex.

1 Introduction

Let G be a digraph. We say that G is 2-*dominating* or that it *dominates pairs* if for every pair $v_1, v_2 \in V(G)$ there exists $u \in V(G)$ such that $\vec{uv_1}, \vec{uv_2} \in E(G)$. More generally, we say G is r-*dominating* or that it *dominates* r-*tuples* if for every r-tuple $v_1, \dots, v_r \in V(G)$ there exists $u \in V(G)$ such that $\vec{uv_1}, \dots, \vec{uv_r} \in E(G)$. We say G dominates pairs (or r-tuples) *uniquely* if the vertex u is unique.

Let g be the (directed) girth of G. If $g \geq 3$ then G is an oriented graph, i.e., for each $u, v \in V(G)$, at most one of the edges $\vec{uv}, \vec{vu}$ lies in $E(G)$. We will be mostly interested in the case when G is an oriented graph. For any vertex $v \in V(G)$, write $\Gamma^+(v) = \{w : \vec{vw} \in E(G)\}$ for the vertices dominated by v.

Myers [10] conjectured that every 2-dominating oriented graph contains a triangle. One of our aims is to give an infinite family of counterexamples to this conjecture. Myers was led to his conjecture by trying to prove a conjecture of Seymour (quoted by Dean and Latka [5]) saying that every oriented graph contains a vertex v such that $|\Gamma^{++}(v)| \geq 2|\Gamma^+(v)|$, where $w \in \Gamma^{++}(v)$ iff w is dominated by some vertex in $\Gamma^+(v) \cup \{v\}$.

The special case of Seymour's conjecture for tournaments, *Dean's conjecture* (see [5]), was proved by Fisher [6], and then a simpler proof was given by Havet and Thomassé [9]. Also, the special

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case of Seymour's conjecture for circulant oriented graphs follows from the Cauchy-Davenport theorem (see [3], [4], [7], [8]) that for $S \subseteq \mathbb{Z}_n$ we have $|S+S| \geq \min\{n, 2|S|-1\}$. A *circulant oriented graph* has vertex set $\mathbb{Z}_n$ and its edges are given by a set $S \subseteq \mathbb{Z}_n \setminus \{0\}$: a vertex a dominates a vertex b iff $b-a \in S$. Our counterexamples to the conjecture of Myers are also circulant oriented graphs, i.e., we shall find sets $S \subseteq \mathbb{Z}_n \setminus \{0\}$ such that $S-S=\mathbb{Z}_n$, $0 \notin S+S$, and $0 \notin S+S+S$. We leave open the question whether $S-S=\mathbb{Z}_n$ implies that $S+S+S+S=\mathbb{Z}_n$, and we do not even know whether there is a k such that if $S-S=\mathbb{Z}_n$ then the k-fold sum of S with itself is the entire $\mathbb{Z}_n$.

Another of our aims in this paper is to show that there are infinitely many uniquely 2-dominating graphs. As we shall see, these are oriented graphs G such that the collection of out-sets $\Gamma^+(v)$, $v \in V(G)$, is the set of lines of a projective plane with point set $V(G)$, and so is the collection of in-sets $\Gamma^-(v)$. Another of the problems we leave open is whether there are triangle-free uniquely 2-dominating graphs. We shall show that the examples we construct all have oriented triangles.

2 Sum sets and difference sets

As stated above, we shall consider circulant digraphs obtained by taking $V(G)=\mathbb{Z}_N$, the integers mod N, and letting $\vec{uv} \in E(G)$ iff $v-u \in S$ for some suitably chosen set $S \subseteq \mathbb{Z}_N \setminus \{0\}$. For the graph to be an oriented graph dominating pairs we need

S1. $S-S=\mathbb{Z}_N$,

S2. $0 \notin S+S$,

where $S \pm S = \{a \pm b : a, b \in S\}$. In general, for the (directed) girth to be $> k$ we need the r-fold sums $S+S+\cdots+S$ not to contain 0 for all $r \leq k$. If S is a set and $n \in \mathbb{Z}$, write $nS = \{nx : x \in S\}$.

Lemma 1 $\min_{S-S=\mathbb{Z}_N} |S+S+S| = o(N)$ *as* $N \to \infty$.

Proof. Let $T=\{2,3,11,14,17,19,21\}$. Then $T-T=\{-19,\dots,19\}\setminus\{\pm 13\}$ and $T+T+T \subseteq \{6,\dots,9\} \cup \{15,\dots,63\} \setminus \{29,58\}$. Pick m minimal so that $N \leq 24(29)^m$ and let $S_m = T + 29T + (29)^2T + \cdots + (29)^mT$. First we prove by induction on m that $S_m - S_m \supseteq \{-12(29)^m, \dots, 12(29)^m\}$. This is clearly true for $m=0$, so assume $m>0$. Now $S_m = 29S_{m-1}+T$, so $S_m - S_m = 29(S_{m-1}-S_{m-1})+(T-T)$. Pick x with $|x| \leq 12(29)^m$. For all such x, we can write $x = 29x' + x''$ with $|x''| \in \{0,\dots,12\} \cup \{14,16\}$ and $|x'| \leq 12(29)^{m-1}$. But $x' \in S_{m-1}-S_{m-1}$ and $x'' \in T-T$. As a consequence, $S_m - S_m$ contains every residue class mod N. Now consider the set $S_m+S_m+S_m$. Let $x \in S_m+S_m+S_m$ and write x in base 29, $x=\sum a_i(29)^i$, $a_i \in \{0,\dots,28\}$. We shall show that it is impossible that $a_i=12$ and $a_{i+1}=1$. Since $x \in S_m+S_m+S_m$, $x=\sum b_i(29)^i$, with $b_i \in T+T+T$. Since $b_i \leq 63$, $\sum_{j=0}^{i-1} b_j(29)^j < 3(29)^i$. Hence b_i must be 12, 11, or 10 mod 29. The only such b_i are 41, 40, and 39. But then $b_{i+1} \equiv 0 \bmod 29$, a contradiction. Since no pair

(a_i, a_{i+1}) can be $(12, 1)$ for any i, the number of elements in $S_m + S_m + S_m$ is $o(3(29)^{m+1}) = o(N)$. Hence if we let S be the set of reductions of elements of S_m mod N, then the number of elements in $S + S + S$ is also $o(N)$. □

Corollary 2 *For all sufficiently large N there is an oriented graph on N vertices which dominates pairs and is (oriented) triangle-free.*

Proof. Take N large enough so that the S given by the previous lemma satisfies $|S+S+S| < \frac{N}{12}$. Clearly $|S + S| \leq |S + S + S|$, so $|2(S + S + S) \cup 3(S + S)| < \frac{N}{6}$, where for a set T and $n \in \mathbb{Z}$, $nT = \{nx : x \in T\}$. Hence there are 6 consecutive elements mod N that do not lie in $2(S+S+S) \cup 3(S+S)$. At least one of these will be divisible by 6 in $\mathbb{Z}_N$, say $6c$, and by replacing S by $S - c$ we can ensure that $0 \notin 2(S + S + S) \cup 3(S + S)$. Then $0 \notin S + S$ and $0 \notin S + S + S$. This then gives an oriented graph as above which has no oriented triangles. □

For $N = 29$ we can take S to be the T defined in Lemma 1. This is the smallest example we know of an oriented triangle-free graph that dominates pairs. There are several other constructions of such graphs. We list three such constructions.

Blowing up vertices.

Take any example of an oriented triangle free graph that dominates pairs (such as the above example on 29 vertices) and replacing one or more vertices by independent sets of vertices to give an example for larger N. This shows that examples exist for all $N \geq 29$. In general the graph constructed will not be a circulant graph.

A simple explicit construction.

Let $n \geq 8$ be an integer and let $S = \{1, 2, \ldots, n-2\} \cup \{n, 2n, 2n+1, 1-2n\}$. If $5n+3 \leq N \leq 6n-5$ then this gives an example on $\mathbb{Z}_N$. Note that such examples exist for all $N \geq 63$.

Base b expansion method.

Choose $b, k > 1$ and let $N = b^k - 1$. For $a \in \mathbb{Z}_N$, considered a as an integer in the range $0, \ldots, N-1$ and write a in base b, $a = \sum_{i=0}^{k-1} a_i b^i$, $a_i \in \{0, \ldots, b-1\}$. Let S be the set of a for which $0 < \sum_{i=0}^{k-1} a_i < k(b-1)/3$. If b and k are sufficiently large then this also gives an example.

Although these constructions are simpler than that given by Lemma 1, we consider Lemma 1 to be of independent interest and pose the following.

Question 1 *Does there exist an N and a set $S \subseteq \mathbb{Z}_N$ such that $S - S = \mathbb{Z}_N$, but $S + S + S + S \neq \mathbb{Z}_N$?*

In case the answer to this question is in the affirmative, is it true that for every $k \geq 3$ there exist an N and a set $S \subseteq \mathbb{Z}_N$ such that $S - S = \mathbb{Z}_N$, but the k-fold sum of S with itself is not the whole of $\mathbb{Z}_N$?

3 Unique domination of pairs

Lemma 3 *Suppose we are given a set of points $P = \{p_1, \ldots, p_n\}$ and lines $\{l_1, \ldots, l_m\}$, $l_i \subseteq P$, with $m \leq n$ such that every pair of points lie in a unique line. Then either*

(a) there is a line containing all the points and all other lines have cardinality ≤ 1; or

(b) there is a line containing $n-1$ points and all other lines consist of one point from this line and the nth point; or

(c) $n = d^2 + d + 1$, the points and lines form a projective plane of order $d \geq 2$.

Proof. If two lines intersect in at least two points then these two points would not lie in a *unique* line. Hence the intersection of two lines contains at most one point. Assume l_1 is the line with the largest number of points, and let $|l_1| = a + 1$. If $a + 1 = n$ then all the other lines can have at most 1 point and we are in case (a). Now assume $a + 1 < n$ so there are some points not in l_1. Then there must be lines that contain a point of l_1 and a point not in l_1. Let $b + 1$ be the maximum size of such a line, say l_2, and assume l_2 intersects l_1 at p. We shall bound the number of lines m. Each pair of points, not equal to p, one from l_1 and one from l_2, specify a unique line, and all these lines are distinct. There are ab such lines none of which contain p. The number of lines containing p is at least $\frac{n-a-1}{b} + 1$ since these partition the $n - 1$ points not equal to p, and apart from l_1 they all contain at most b points not equal to p. Hence

$$ab + 1 + \tfrac{n-a-1}{b} \leq m \leq n. \tag{1}$$

Rearranging gives $a(b^2 - 1) \leq (n - 1)(b - 1)$. Assume now that $b > 1$. Then

$$n - 1 \geq a(b + 1). \tag{2}$$

The number of lines other than l_1 going through each point of l_1 is at least $\frac{n-a-1}{b}$, and these lines are all distinct. Hence

$$\tfrac{n-a-1}{b}(a + 1) + 1 \leq m \leq n. \tag{3}$$

Thus

$$(n - 1)(a + 1 - b) \leq a(a + 1). \tag{4}$$

Substituting inequality (2) into (4) gives

$$(n - 1)(a + 2)a - (n - 1)^2 \leq a^2(a + 1) \tag{5}$$

or, rearranging,

$$(n - 1 - a)(n - 1 - a - a^2) \geq 0. \tag{6}$$

The case $n-1-a \leq 0$ implies that l_1 contains all the points. Hence we may assume $n-1-a > 0$. Thus

$$n - 1 \geq a(a + 1). \tag{7}$$

This together with equation (4) gives $b = a$ and $n = 1 + a + a^2$. In fact, there are at most a lines other than l_1 through p, so all lines through p must have $a + 1$ points. Since every point in now on a line with $a + 1$ points, we see that every line has $a + 1$ (or 0) points. If we had two non-intersecting lines with $a + 1$ points, then by considering lines meeting one point of the first and one point of the second, we would have a total of at least $(a + 1)^2 > n$ lines, a contradiction. Hence every two lines intersect in a single point, and the set of lines and points form a projective plane.

The only remaining case is when $b \le 1$, so every line intersecting l_1 has at most 2 points. Hence for each pair of points, one in l_1 and one outside l_1, there is a unique line through the pair, and all such lines are distinct. This gives a total of $(a+1)(n-a-1)+1$ lines. Thus $(a+1)(n-a-1) \le n-1$, so

$$a(n - 1) \le a(a + 1). \tag{8}$$

Hence $n \le a + 2$ and we are in case (b). □

Corollary 4 *If G is an oriented graph that dominates pairs uniquely and $|V(G)| > 1$ then the sets $\Gamma^+(v)$ form the lines of a projective plane on $V(G)$.*

Proof. Every pair of points is uniquely dominated, so lies in a unique line $\Gamma^+(v)$. The number of lines is the same as the number of points. Hence by Lemma 3 we either have a projective plane, or one of the two special cases listed in that lemma. It is easy to see that the two special cases cannot give rise to an oriented graph. □

It remains to show that such oriented graphs exist. For this we consider the known projective planes, given by the lines in a three-dimensional vector space over a finite field.

Theorem 5 *For all $q = p^n$, p prime, $n \ge 1$, there exists an oriented graph of order $q^2 + q + 1$ that dominates pairs uniquely.*

Proof. Let $\mathbb{F}_q$ be the field with q elements. Let $\mathbb{F}_{q^3}$ be the (unique) cubic extension of $\mathbb{F}_q$. Then we can regard $\mathbb{F}_{q^3}$ as a 3-dimensional vector space over $\mathbb{F}_q$, and we can therefore regard the projective plane over $\mathbb{F}_q$ as $\mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$, where $\mathbb{F}_{q^3}^\times$ and $\mathbb{F}_q^\times$ are the set of non-zero elements of $\mathbb{F}_{q^3}$ and $\mathbb{F}_q$ respectively. The lines of this projective plane correspond to 2-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^3}$. Recall that the trace map $\mathrm{Tr}\colon \mathbb{F}_{q^3} \to \mathbb{F}_q$ is a surjective $\mathbb{F}_q$-linear map. For $\alpha \in F_{q^3}^\times$ (to be determined) let G_α be the following graph.

G1. $V(G_\alpha) = \mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$,

G2. $E(G_\alpha) = \{\vec{uv} : \mathrm{Tr}(\alpha v/u) = 0\}$,

where the condition $\mathrm{Tr}(\alpha v/u) = 0$ is independent of the choice of representatives of u, v in $\mathbb{F}_{q^3}^\times$. The set $\Gamma^+(u)$ corresponds to a 2-dimensional subspace of $\mathbb{F}_{q^3}$, so is a line in the projective space.

If the lines given by u_1 and u_2 are the same, then the linear maps $v \to \mathrm{Tr}(\alpha v/u_i)$ have the same kernel. But this implies the maps are proportional, $\mathrm{Tr}(\alpha v/u_2) = \lambda \mathrm{Tr}(\alpha v/u_1) = \mathrm{Tr}(\lambda \alpha v/u_1)$ for all v. By letting v run over a basis for $\mathbb{F}_{q^3}$, we see that $\alpha/u_2 = \lambda\alpha/u_1$, so $u_1 = \lambda u_2$, and u_1 and u_2 give the same vertex of G_α. The only remaining conditions concern the girth. These are implied by the following condition

C1. If $\mathrm{Tr}(x) = \mathrm{Tr}(y) = 0$ then $xy \neq \alpha^2$.

To see this, take $x = y = \alpha$. Then $\mathrm{Tr}(\alpha v/v) \neq 0$ so G_α contains no loops. If $u, v \in V(G_\alpha)$, take $x = \alpha u/v$, $y = \alpha v/u$. Then the condition shows that we cannot have both $\vec{uv}$ and $\vec{vu}$ in $E(G)$. The result now follows from the following lemma. □

Lemma 6 *For every $q = p^n$, p prime, $n \geq 1$, there exists $\alpha \in \mathbb{F}_{q^3}$ such that condition C1 above holds.*

Proof. The kernel of the trace map is a 2-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^3}$. Let $\{\eta, \eta'\}$ be a basis for this subspace and let $\gamma = \eta'/\eta$. Now $\gamma \notin \mathbb{F}_q$, so $\mathbb{F}_q(\gamma) = \mathbb{F}_{q^3}$ and $\{1, \gamma, \gamma^2\}$ is a basis for $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$.

The map $\mathbb{F}_q^\times \to \mathbb{F}_q; k \mapsto k + 1/k$ is not surjective since $|\mathbb{F}_q^\times| < |\mathbb{F}_q|$. Hence there is an element $c \in \mathbb{F}_q$ not of the form $k+1/k$. Let $\beta = \eta^2 + c\eta\eta' + \eta'^2$. Then β is not the product of two trace-free numbers. Indeed, if $(a_1\eta + a_2\eta')(b_1\eta + b_2\eta') = \beta$ then $1 + c\gamma + \gamma^2 = (a_1b_1) + (a_1b_2 + a_2b_1)\gamma + (a_2b_2)\gamma^2$. But since $\{1, \gamma, \gamma^2\}$ is a basis over $\mathbb{F}_q$, we get

$$a_1b_1 = 1, \qquad a_2b_2 = 1, \qquad a_1b_2 + a_2b_1 = c. \tag{9}$$

This implies $c = k + 1/k$ where $k = a_1/a_2 \in \mathbb{F}_q$, a contradiction.

If λ is not a square in $\mathbb{F}_q$ then it is not a square in $\mathbb{F}_{q^3}$ (otherwise $\mathbb{F}_q(\sqrt{\lambda})$ would be a quadratic extension of $\mathbb{F}_q$ lying in $\mathbb{F}_{q^3}$). Hence some element of the form $\lambda\beta$, $\lambda \in \mathbb{F}_q^\times$, will be a perfect square in $\mathbb{F}_{q^3}$, since if β is not a perfect square, we can take λ to be a non-square in $\mathbb{F}_q$. Now choose α so that $\alpha^2 = \lambda\beta$. □

If $q \equiv 2 \bmod 3$ we can take $\alpha = 1$ in the lemma. To see this, we note that the trace is the sum of the conjugates, $\mathrm{Tr}(x) = x + x^q + x^{q^2}$. If $\mathrm{Tr}(x) = \mathrm{Tr}(1/x) = 0$ then $x + x^q + x^{q^2} = x^{q^2} + x^{q^2-q+1} + x = 0$. Thus $x^{q^2-2q+1} = 1$. Thus the order of x in the group $\mathbb{F}_{q^3}^\times$ divides $\gcd(q^2 - 2q + 1, q^3 - 1) = (q-1)\gcd(q-1, q^2+q+1) = q - 1$. But then $x^q = x$, so $x \in \mathbb{F}_q$. But then $\mathrm{Tr}(x) = 3x \neq 0$, a contradiction.

It is worth noting that the graphs obtained above have a cyclic automorphism. Indeed, $\mathbb{F}_{q^3}^\times$ is a cyclic group under multiplication of order $q^3 - 1$, so $\mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$ has the structure of a cyclic group of order $q^2 + q + 1$. The condition $\mathrm{Tr}(\alpha u/v) = 0$ is just the condition that the difference in this cyclic group lies in a certain set S, so the graph can be described as a circulant graph on $\mathbb{Z}_N$ where $N = q^2 + q + 1$ and $|S| = q + 1$

4 Unique domination of n-tuples

Lemma 7 *Suppose we are given a set of points $P = \{p_1, \dots, p_n\}$ and lines $\{l_1, \dots, l_m\}$, $l_i \subseteq P$, with $m \le n$, $n \ge r \ge 3$, such that every r-tuple of points lie in a unique line. Then either*

(a) there is a line containing all the points and all other lines have cardinality $< r$; or

(b) $m = n = r + 1$ and the lines consist of all subsets of P of size $n - 1$.

Proof. If there is a line containing every point then we are in case (a), and this must occur if $r = n$, so assume $r < n$ and some point, p_n say, does not lie in every line. Let $l_1, \dots, l_k$ be the lines containing p_n and $l_{k+1}, \dots, l_m$ the lines not containing p_n. Each $(r-1)$-tuple of points in $\{p_1, \dots, p_{n-1}\}$ lies in an l_i with $i \le k$ since adding p_n to this $(r-1)$-tuple gives an r-tuple which lies in a line l_i, and $i \le k$ since this line contains p_n. If the $(r-1)$-tuple lies in two such lines, we would have two lines containing this $(r-1)$-tuple and p_n. Thus every $(r-1)$-tuple lies in a unique line $l_1, \dots, l_k$. Since $k \le n - 1$ we can apply induction on r. If one of the lines l_i, $i \le k$ contains all the points of $P \setminus \{p_n\}$ then it contains all of P and we are in case (a). In all other cases in Lemma 3 or by induction on r from Lemma 7, $k = n - 1$. Hence there is only one line l_n not containing p_n. Pick any point $p_i \ne p_n$ and some $(r-1)$-tuple of points containing p_i but not p_n. This $(r-1)$-tuple must lie in some unique line l_j, $j \le k$. Pick a point p_h not in l_j. Then the r-tuple obtained by adding p_h can only lie in l_n and so l_n contains p_i. Thus $l_n = P \setminus \{p_n\}$ and every other line can contain at most $r - 1$ points from $P \setminus \{p_n\}$. There are $\binom{n-1}{r-1}$ r-tuples containing p_n, but each line $l_1, \dots, l_{n-1}$ can only contain one of these. Thus $\binom{n-1}{r-1} \le n - 1$. Since $3 \le r < n$ we have $r = n - 1$ and the lines consist of all $(n-1)$-tuples of points. □

Theorem 8 *For $r \ge 3$, the only directed graph with $|V(G)| \ge r$ that dominates r-tuples uniquely is the complete digraph on $r + 1$ vertices.*

Proof. The lines $\Gamma^+(v)$ satisfy the conditions of Lemma 7. However, $v \notin \Gamma^+(v)$ so we cannot be in case (a). Thus we are in case (b) with $|V(G)| = r + 1$ and $|\Gamma^+(v)| = r$ for all r. Hence there is a directed edge from v to every other vertex, and the graph is the complete digraph on $r + 1$ vertices. □

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