PERCOLATION IN VORONOI TILINGS

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Abstract. We consider a percolation process on a random tiling of $\mathbb{R}^d$ into Voronoi cells based on points of a realization of a Poisson process. We prove the existence of a phase transition as the proportion p of open cells is varied and provide estimates for the critical probability p_c. Specifically, we prove that for large d, $2^{-d}(9d\log d)^{-1} \le p_c(d) \le C2^{-d}\sqrt{d}\log d$.

1. Introduction

We consider a percolation process on a random tiling of $\mathbb{R}^d$ (for $d > 1$) induced by a labelled Poisson process. Specifically, one takes a parameter $p \in [0,1]$ and a Poisson process on $\mathbb{R}^d$ with density constant and equal to 1. A realization of the process consists (almost surely) of a countably infinite subset of $\mathbb{R}^d$. For a given realization, the points in it will be called *Poisson points*. Each Poisson point is independently labelled as *open* with probability p or *closed* with probability $1-p$. Corresponding to each Poisson point x, we construct its *Voronoi cell* which consists of those points of $\mathbb{R}^d$ that are closer to x than to any other Poisson point. We note that almost surely the realizations of the Poisson process are discrete so that the Voronoi cells are convex open sets. Since there are countably many points in a realization of the Poisson process, it is clear that the points that are equidistant from two or more Poisson points have Lebesgue measure 0 so that the Voronoi cells form a partition of $\mathbb{R}^d$ up to sets of measure 0.

One can show that for almost every point of the Poisson process, the Voronoi cells are bounded and have finitely many $(d-1)$-dimensional faces. Two Voronoi cells are said to be neighbours if they share a $(d-1)$-dimensional face. A collection of cells is said to be connected if for any pair of cells, there is a path from one to the other consisting of neighbouring cells that belong to the collection. Given any collection of cells, its connected components will (as usual) be called *clusters*.

We then define a percolation on the Voronoi cells as follows: each cell is labelled as open or closed according to whether the Poisson point that it contains is open or closed. The labelled collection of cells is said to *percolate* if there is an infinite cluster of open cells. We call the labelled random division of $\mathbb{R}^d$ into Voronoi cells the *Voronoi percolation process*. It

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is a common situation when studying percolation that there exists a constant $p_c \in [0,1]$ (the *critical probability*) so that for $p < p_c$, the probability that there is percolation is 0, whereas for $p > p_c$, the probability that there is percolation is 1. In the case where $0 < p_c < 1$, the process is said to have a *phase transition*.

Straightforward monotonicity and independence arguments show that there exists a critical probability. Indeed, the event that there is percolation is a tail event in the relevant filtration of σ-algebras so that for a given parameter p, percolation occurs with probability 0 or 1. If $p' > p$ then a coupling argument shows that if there is percolation with probability 1 for parameter p, then there is percolation with probability 1 for parameter p'. In order to show that there is a phase transition, it is therefore sufficient to find a lower bound (a number $p_- > 0$ such that the system with parameter p_- does not percolate) and an upper bound (a number $p_+ < 1$ so that the system with parameter p_+ does percolate). In this case, of course, one sees that $p_- \le p_c \le p_+$.

We are now ready to state the main theorem.

Theorem 1. *The Voronoi percolation process in $\mathbb{R}^d$ for d sufficiently large has a critical probability $p_c(d)$ satisfying*

$$2^{-d}(9d\log d)^{-1} \le p_c(d) \le C2^{-d}\sqrt{d}\log d$$

for a constant C independent of the dimension.

Two halves of the theorem (the upper and lower bounds) are proved in the next two sections. The methods, while different, have a key feature in common. Namely, in each case, the proof proceeds by comparing the Voronoi percolation process with an independent percolation process on a lattice. To establish the lower bound, the Voronoi percolation process is compared with a site percolation process in $\mathbb{Z}^d$ that can be easily seen not to percolate, whereas to establish the upper bound, the Voronoi percolation is compared with an oriented percolation process in $\mathbb{Z}^2$, in which percolation has already been established. The key difficulty with an approach of this type is to establish sufficient independence as it may be seen that in the Voronoi percolation process, there necessarily exist arbitrarily large Voronoi cells and hence the process has long-range dependence. In the proof below, we present a surprisingly simple solution to this difficulty, which we hope will find application in other problems.

For related work, the reader is referred to work of Freedman [**?**] on Voronoi percolation in the projective plane, Penrose [**?**] on connectivity and minimum degree of geometric graphs and Vahidi-Asl and Wierman [**?**, **?**] on first passage Voronoi percolation in the plane. Benjamini and Schramm [**?**] consider various percolation models in the hyperbolic plane including Voronoi percolation. That paper contains a lower bound argument similar to the one that we present here.

2. The Lower Bound

Starting from a Voronoi percolation process on $\mathbb{R}^d$, we introduce a site percolation process on $\mathbb{Z}^d$, which we use to prove the absence of percolation in the original Voronoi process. The

$\mathbb{Z}^d$ process fails to have the usual independence between sites, but has instead a dependence that is of finite range.

To construct the site percolation process on $\mathbb{Z}^d$, we start off with a realization of the Voronoi process and divide $\mathbb{R}^d$ into cubes of side R. Each cube corresponds to a vertex in $\mathbb{Z}^d$ and two such cubes are *adjacent* if their closures intersect, which is equivalent to sites being diagonally adjacent ($\|.\|_\infty$-distance 1 apart) in $\mathbb{Z}^d$. For a set C, $B_r(C)$ will denote $\{x\colon \inf_{y\in C} \|x-y\|_2 \le r\}$. We will write $B_r(x)$ for $B_r(\{x\})$.

A cube C in $\mathbb{R}^d$ (or the corresponding vertex in $\mathbb{Z}^d$) is said to be *open* if either

(1) there exists a point $x \in C$ such that $B_{R/6}(x)$ contains no point of the underlying Poisson process; or
(2) there exists a curve γ in $B_{R/6}(C)$ such that $\gamma(0) \in \partial C$, $\gamma(1) \in \partial B_{R/6}(C)$, and $\gamma(t)$ belongs to an open Voronoi cell for all $t \in [0, 1]$.

Lemma 2. *The openness of a cube C is determined by the restriction of the labelled Poisson process to $B_{R/2}(C)$.*

Proof. The cube C is open if (1) holds or if (2') holds, where (2') is the condition that (1) fails but (2) holds.

Clearly whether or not condition (1) is satisfied is determined by the restriction of the Poisson process to $B_{R/6}(C)$. It is sufficient to show that (2') is determined by the restriction of the Poisson process to $B_{R/2}(C)$. Given that (1) fails, every point of $B_{R/6}(C)$ is within $R/3$ of a point of the Poisson process. The Voronoi cells restricted to $B_{R/6}(C)$ are therefore determined by the restriction of the Poisson process to $B_{R/2}(C)$. The information in the restriction of the labelled Poisson process to $B_{R/2}(C)$ is thus sufficient to determine whether or not (2') holds. □

It follows from Lemma 2 that openness of two sites in the $\mathbb{Z}^d$ process are independent provided that they are not (diagonally) adjacent.

Lemma 3. *The probability of percolation in the $\mathbb{Z}^d$ process defined above is bounded below by the probability of percolation in the Voronoi process.*

Proof. Given a configuration of Voronoi cells with an infinite open cluster, there is an unbounded curve γ in $\mathbb{R}^d$ such that $\gamma(t)$ lies in the cluster for all t. If $\gamma(t) \in \bar{C} \cap \bar{C}'$, then C and C' are adjacent. Also if $\gamma(t) \in C$ then C satisfies condition (2) above so it is open in the $\mathbb{Z}^d$ process. The lemma follows. □

Hence to show that the Voronoi process does not percolate, it is sufficient to show that the site process does not percolate.

Lemma 4. *Let p be the probability that a given cube of side R is open. If $p < 9^{-d}$ then there is almost surely no percolation in the $\mathbb{Z}^d$ process and hence no percolation in the Voronoi process.*

Proof. We say that a cube C of side R in the lattice is accessible from the origin if there is a sequence of cubes $0 = C_0, C_1, \ldots, C_n = C$ in the lattice that are all open and such that C_i is adjacent to C_{i+1}. We take such a sequence with minimal length and note that C_i is adjacent to C_j if and only if $|i-j| = 1$. Such a sequence is an open induced path in the lattice.

We need to show that almost surely the number of cubes accessible from the origin is finite. We show that almost surely there exists an N such that there are no open induced paths of length N. The number of induced paths of length N is bounded above by 3^{Nd} and the probability that an induced path of length N is open is bounded above by $p^{N/2}$ since taking every second cube on the path, it is independently open with probability p. Accordingly, the expected number of open induced paths of length N is bounded above by $(9^d p)^{N/2}$. Since $9^d p < 1$, the conclusion follows. $\square$

We now turn to showing that the probability that a cube is open is less than 9^{-d}. Let the density of open points of the Poisson process be given by $\rho = 2^{-d}/(9d \log d)$. Let r_d be the radius of a d-dimensional sphere of volume 1 (so that $r_d = (d/2)!^{1/d}/\sqrt{\pi} \sim \sqrt{d/2\pi e}$). Set $R = r_d d^{1.1}$.

Lemma 5. *Let A_1 be the event that each point in $B_{R/2}(C)$ has a point of the Poisson process within $r_d(4d\log d)^{1/d}$. Then $\mathbb{P}(A_1^c) = o(9^{-d})$.*

Proof. Let $x \in B_{R/2}(C)$. The probability that x has no point of the Poisson process within $r_d(3d\log d)^{1/d}$ is given by $\exp(-3d\log d) = d^{-3d}$. Let $\epsilon = r_d((4d\log d)^{1/d} - (3d\log d)^{1/d}) > 1/(15\sqrt{d})$. Now take an ϵ-net of points x covering $B_{R/2}(C)$ (this can be achieved with at most $(2R/(\epsilon/\sqrt{d}) + 1)^d$ points). The probability that there is a point of the ϵ-net with no point of the Poisson process within $r_d(3d\log d)^{1/d}$ is bounded above by $(2R\sqrt{d}/\epsilon + 1)^d d^{-3d} = O(d^{2.6}d^{-3})^d = o(9^{-d})$.

Given that each point of the ϵ-net has a point within $r_d(3d\log d)^{1/d}$, it follows that each point of $B_{R/2}(C)$ has a point within $r_d(3d\log d)^{1/d} + \epsilon = r_d(4d\log d)^{1/d}$. Hence $\mathbb{P}(A_1^c) = o(9^{-d})$ as required. $\square$

We observe that given that A_1 holds, each Voronoi cell has radius (about the Poisson point) bounded above by $r_d(4d\log d)^{1/d}$ and hence diameter bounded above by $2r_d(4d\log d)^{1/d}$. We also note that $r_d(4d\log d)^{1/d} = o(R)$ as $d \to \infty$ so condition (1) holds with probability $o(9^{-d})$.

Lemma 6. *Let A_2 be the event that each Voronoi cell in $B_{R/6}(C)$ has at most $2^d(8d\log d)$ neighbors and let A_3 be the event that there are at most $2(2R)^d$ Voronoi cells intersecting ∂C. Then $\mathbb{P}(A_2^c) = o(9^{-d})$ and $\mathbb{P}(A_3^c) = o(9^{-d})$.*

Proof. Since we showed above that $\mathbb{P}(A_1^c) = o(9^{-d})$, it is enough to show that $\mathbb{P}(A_2^c \cap A_1)$ and $\mathbb{P}(A_3^c \cap A_1)$ are both $o(9^{-d})$. Given that we have a configuration belonging to A_1, it follows that if $x \in B_{R/6}(C)$ is a point of the Poisson process, all points of the Poisson process whose Voronoi cells are adjacent to the Voronoi cell of x are at a distance at most $2r_d(4d\log d)^{1/d}$ from x.

We now estimate the probability that there is a ball of radius $2r_d(4d\log d)^{1/d}$ containing more than $2^d(8d\log d)$ points of the Poisson process. First in a given ball of radius $2r_d(5d\log d)^{1/d}$, the probability that there are more than $2^d(8d\log d)$ points of the Poisson process is estimated above by 3 times the probability that there are exactly $\lceil 2^d(8d\log d)\rceil$ Poisson points in the ball. This in turn is bounded above by $2^d((5/8)^8e^3)^{2^d(d\log d)}$ or more crudely by $(1/2)^{2^d}$. As in Lemma 5, we take an ϵ-net of such balls where $\epsilon = r_d((5d\log d)^{1/d} - (4d\log d)^{1/d})$ so that as before, the probability that any of them contains more than $2^d(8d\log d)$ points of the Poisson process is bounded above by $O(d^{2.6})^d(1/2)^{2^d} = o(9^{-d})$. This is sufficient to ensure that the probability that there exists a $2r_d(4d\log d)^{1/d}$ ball containing more than $2^d(8d\log d)$ Poisson points is $o(9^{-d})$ so $\mathbb{P}(A_2^c) = o(9^{-d})$.

By a similar estimate on a Poisson random variable to the one above, there are at most $2(2R)^d$ points of the Poisson process in $B_{R/2}(C)$ with probability at least $1 - 2(e/4)^{(2R)^d}$, so that if A_1 holds then the number of cells meeting ∂C is bounded by the number of points of the Poisson process inside $B_{R/2}(C)$, which is bounded by $2(2R)^d$ with probability $1 - o(9^{-d})$. □

Lemma 7. *Let A be the event that the cube C of side R is open. Then $\mathbb{P}(A) = o(9^{-d})$.*

Proof. By the previous two lemmas, we may restrict our attention to configurations belonging to $A_1 \cap A_2 \cap A_3$. Since by the earlier observation in this case all cells have diameter bounded above by $2r_d(4d\log d)^{1/d}$, Condition (1) fails automatically. For condition (2) to hold, there needs to be a path of open Voronoi cells consisting of at least $(R/6)/(2r_d(4d\log d)^{1/d})$ tiles. We estimate that $(R/6)/(2r_d(4d\log d)^{1/d}) > d^{1.1}/30$. It follows that $\mathbb{P}(A)$ is bounded above by the probability that there exists a path of open tiles of length $d^{1.1}/30$ starting from ∂C.

Writing E for the expected value of the number of open paths of length $d^{1.1}/30$ times the indicator function of $A_1 \cap A_2 \cap A_3$,

$$E \le 2(2R)^d \left(\frac{2^d(8d\log d)}{2^d(9d\log d)}\right)^{d^{1.1}/30} = O(d^{3d}(8/9)^{d^{1.1}/30}) = o(9^{-d}).$$

Since the expected value is an upper bound for the probability of existence, we see that $\mathbb{P}(A \cap A_1 \cap A_2 \cap A_3) = o(9^{-d})$ and hence $\mathbb{P}(A) = o(9^{-d})$ as required. □

3. The Upper Bound

We shall bound the probability of the Voronoi process percolating by comparing it with an oriented percolation process in $\mathbb{N}^2$. However, to get the necessary independence, we shall use three dimensions in the construction.

The basic idea is to represent the edges and vertices of $\mathbb{Z}^3$ as cylinder regions in $\mathbb{R}^d$. We shall consider the edges of $\mathbb{Z}^3$ to be of unit length, oriented in the positive x, y, or z directions. The edges of $\mathbb{Z}^3$ will be represented by rectangular shaped regions L_{S_x,S_y,S_z} that we shall call *links*. These will be chosen so that links are well separated when the corresponding edges of $\mathbb{Z}^3$ do not share a vertex. The vertices (x, y, z) of $\mathbb{Z}^3$ will correspond to slightly more complicated regions $B_{x,y,z}$ that consist of the ends of the links leading into (x, y, z), and are adjacent to the links leading out of (x, y, z). We try to obtain percolation by joining an open point in $B_{x,y,z}$ with an open point in adjacent B's by a sequence of open cells in the links. Eventually we shall compare this process to a 2-dimensional oriented percolation by 'collapsing' the y and z coordinates into one coordinate $y + z$. Using 3-dimensions in the construction allows us to maintain independence when there are several paths from the origin to a given $(x, y) \in \mathbb{Z}^2$ since the paths will actually end up in distinct sites (x, y', z') with $y' + z' = y$.

First we prove a simple geometric lemma.

Assume that $d > 3$ and fix three of the dimensions. For $S_x, S_y, S_z \subseteq \mathbb{R}$, denote by L_{S_x,S_y,S_z} the 'cylinder' set of points with first coordinate lying in S_x, second coordinate lying in S_y, third coordinate lying in S_z, and all other coordinates unrestricted.

Lemma 8. *Assume d is sufficiently large. Then the volume of the sphere $B_{r_d}(0)$ lying in the region $L_{[\delta,\delta+1),[0,\infty),[0,\infty)}$ is bounded below by some constant $c > 0$ independently of d and δ for all δ with $0 \le \delta \le 1$.*

Proof. The volume of the specified region is given exactly by

$$\frac{1}{4}\int_{\delta}^{\delta+1} (r_d^2 - x^2)^{(d-1)/2} r_{d-1}^{-(d-1)}\, dx.$$

Since $\delta + 1 \le 2$ we can bound this below by

$$\frac{1}{4}((r_d^2 - 4)/r_{d-1}^2)^{(d-1)/2}.$$

However $r_d^2 \sim d/2\pi e$, so this is bounded below by $\frac{1}{4}(1 - O(1/d))^{(d-1)/2}$ which is bounded below by a constant $c > 0$ independently of d and δ. □

Now we show that percolation is reasonably likely along a single link.

Lemma 9. *Assume the density of open points is ρ and let $0 < \eta < 1$. Let P be a fixed point in $B = L_{[-2\eta,0),[0,4r_d),[0,4r_d)} \cup L_{[0,4r_d),[-2\eta,0),[0,4r_d)} \cup L_{[0,4r_d),[0,4r_d),[-2\eta,0)}$. Then with probability at least $1 - \lceil 4r_d/\eta \rceil \exp(-2^d\eta^d\rho c)$ there is a sequence $P_0, \dots, P_n$ of points with $n \le \lceil 4r_d/\eta \rceil$, $P_0 = P$, P_i an open point of the process lying in $L_{[0,8r_d),[0,4r_d),[0,4r_d)}$ for $0 < i \le n$, and $P_n \in L_{[8r_d-2\eta,8r_d),[0,4r_d),[0,4r_d)}$, with $d(P_i, P_{i+1}) \le 2\eta r_d$ for $i = 0, \dots, n-1$.*

Proof. Assume first that $P \in L_{[-2\eta,0),[0,4r_d),[0,4r_d)}$. Using Lemma 8 with $\delta = 1$ we see that with probability $1 - \exp(-2^d\eta^d\rho c)$ there is an open point within distance $2\eta r_d$ of P with x-coordinate between 2η and 4η more than for P and the changes in y and z coordinate of the right signs to keep these coordinates between 0 and $4r_d$. Repeating this process at most $\lceil 4r_d/\eta \rceil - 1$ times we get a point inside $L_{[8r_d-4\eta,8r_d),[0,4r_d),[0,4r_d)}$. By taking one more point using a suitable choice of δ in Lemma 8 if necessary, we can get the last point with x-coordinate in $[8r_d - 2\eta, 8r_d)$ and we are done. The probability of failure is at most $n\exp(-2^d\eta^d\rho c)$ where $n \le \lceil 4r_d/\eta \rceil$. For the cases when $P \in L_{[0,4r_d),[-2\eta,0),[0,4r_d)}$ or $L_{[0,4r_d),[0,4r_d),[-2\eta,0)}$ we take the first step in the y or z direction and the remaining steps in the x direction as above. □

Lemma 10. *There exists a constant C such that for $\rho > C\,2^{-d}\sqrt{d}\,\log d$ the Voronoi process with open point density ρ has infinite clusters with probability 1.*

Proof. We now construct a site percolation in $\mathbb{N}^2$ which has less chance of percolating than the original Voronoi percolation. Each vertex $(x, y) \in \mathbb{N}^2$ will correspond to a region

$$\begin{aligned} B_{x,y',z'} &= B + (8r_d x, 8r_d y', 8r_d z') \\ &= L_{[8r_d x-2\eta,8r_d x),[8r_d y',8r_d y'+4r_d),[8r_d z',8r_d z'+4r_d)} \\ &\quad \cup L_{[8r_d x,8r_d x+4r_d),[8r_d y'-2\eta,8r_d y'),[8r_d z',8r_d z'+4r_d)} \\ &\quad \cup L_{[8r_d x,8r_d x+4r_d),[8r_d y',8r_d y'+4r_d),[8r_d z'-2\eta,8r_d z')} \end{aligned}$$

where B is the region defined in the statement of Lemma 9 and $y' = y'(x, y)$, $z' = z'(x, y)$ are integers to be determined with $y' + z' = y$. An edge from (x, y) to $(x + 1, y)$ will correspond to the region

$$L_{x^+,y',z'} = L_{[8r_d x,8r_d x+8r_d),[8r_d y',8r_d y'+4r_d),[8r_d z',8r_d z'+4r_d)}.$$

An edge from (x, y) to $(x, y + 1)$ will correspond to one of the regions

$$L_{x,y'^+,z'} = L_{[8r_d x,8r_d x+4r_d),[8r_d y',8r_d y'+8r_d),[8r_d z',8r_d z'+4r_d)}$$

or

$$L_{x,y',z'^+} = L_{[8r_d x,8r_d x+4r_d),[8r_d y',8r_d y'+4r_d),[8r_d z',8r_d z'+8r_d)}.$$

These regions will be our links. Note that regardless of the choices of y' or z', the links corresponding to non-incident edges in $\mathbb{N}^2$ are disjoint, separated by a distance of at least $4r_d$, and $B_{x,y',z'}$ borders each of the links $L_{x^+,y',z'}$, $L_{x,y'^+,z'}$, and L_{x,y',z'^+}.

The idea is to identify sites (x, y) of $\mathbb{N}^2$ with regions of the form $B_{x,y',z'}$ and bonds of $\mathbb{N}^2$ with links joining these regions. A bond will be *open* if we can construct a sequence of points P_i along the corresponding link as in Lemma 9. We shall choose $y' = y'(x, y)$ and $z' = z'(x, y)$ so as to ensure that certain links are well separated. This will give us independence of the corresponding bonds in $\mathbb{N}^2$.

We now construct the process on $\mathbb{N}^2$ by induction on the level $k = x+y$. For $k = x = y = 0$, set $x' = y' = 0$ and pick an open point $P_{0,0}$ of the Poisson process in $B_{0,0,0}$. Since this region has infinite volume for $d > 3$, this will be possible with probability 1. Now suppose for all x, y with $x + y = k$ we have defined $y'(x, y)$ and $z'(x, y)$ and open points of the Poisson process $P_{x,y} \in B_{x,y',z'}$. Declare the edge (x, y) to $(x + 1, y)$ in $\mathbb{N}^2$ to be *open* if there is a sequence of points $P_{x,y} = P_0, P_1, \ldots, P_n$ in $L_{x^+,y',z'}$ as in Lemma 9 and the spheres with diameters P_iP_{i+1} do not contain any closed points of the Poisson process for $i = 0, \ldots, n-1$. If P_0 is itself an open point of the Poisson process then this last condition ensures that any point on the line joining P_i to P_{i+1} is closer to an open point than a closed point, and hence this line passes through open Voronoi cells only. Thus the Voronoi cells centered on P_0 and P_n lie in the same connected component of open cells and $P_n \in B_{x+1,y',z'}$. Moreover, the event that this edge is open depends only on the restriction of the Poisson process to $B_{\eta r_d}(L_{x^+,y',z'}) \cup B_{\eta r_d}(P_0P_1)$. Define $\delta_{x,y} = 1$ if $y'(x-1, y+1) = y'(x, y)$, otherwise define $\delta_{x,y} = 0$. Declare the edge from (x, y) to $(x, y + 1)$ open if we can find a similar sequence of points in $L_{x,y'^+,z'}$ (if $\delta_{x,y} = 1$)

or L_{x,y',z'^+} (if $\delta_{x,y}=0$). The choice of $\delta_{x,y}$ ensures that the link used does not meet the link $L_{(x-1)^+,y'(x-1,y+1),z'(x-1,y+1)}$. Hence the links used corresponding to distinct (x,y) at level k are always separated by a distance of at least $4r_d > 2\eta r_d$.

The openness of the edges in $\mathbb{N}^2$ is dependent on the choice of the $P_{x,y}$ which will depend on the open points of the Poisson process in the links at previous levels, and on the closed points in the ηr_d-neighborhoods of these links. Conditional on these choices two edges from level k to level $k+1$ are however independent provided they start from different points (x,y) at level k. This is due to the fact that the corresponding regions they depend on are disjoint.

Now choose $P_{x,y}$ and y' and z' at level $k+1$ according to the following rules.

- If $(x-1,y)\in\mathbb{N}^2$ is accessible from the origin in the $\mathbb{N}^2$ percolation process and if the edge $(x-1,y)$ to (x,y) is open, choose $P_{x,y}$ to be the P_n of the sequence of points chosen for this edge above. Define $y'(x,y)=y'(x-1,y)$ and $z'(x,y)=z'(x-1,y)$ so that $P_{x,y}\in B_{x,y',z'}$.
- Otherwise, if $(x,y-1)$ is accessible from the origin and if the edge $(x,y-1)$ to (x,y) is open, choose $P_{x,y}$ to be the P_n of the sequence of points chosen for this edge. Define $y'(x,y)=y'(x,y-1)+\delta_{x,y-1}$ and $z'(x,y)=z'(x,y-1)+1-\delta_{x,y-1}$ so that $P_{x,y}\in B_{x,y',z'}$.
- If neither of the above conditions holds, choose $P_{x,y}$, $y'(x,y)$, and $z'(x,y)$ arbitrarily so that $P_{x,y}\in B_{x,y',z'}$ is an open point of the Poisson process and $y'+z'=y$.

It is clear that if the $\mathbb{N}^2$ process percolates then the Voronoi process also percolates since we can join up the sequences of points $P_0,\dots,P_n$ from each edge in an infinite path in $\mathbb{N}^2$ to get a sequence of open connected Voronoi cells. It remains to show that the $\mathbb{N}^2$ process can be compared with an independent process which we know does percolate.

In the $\mathbb{N}^2$ process the edges from (x,y) to $(x+1,y)$ and $(x,y+1)$ are both open with conditional probability at least $1-2\lceil 4r_d/\eta\rceil(\exp(-2^d\eta^d\rho c)+\eta^d)$ regardless of the state of the edges at previous levels. Here η^d bounds the probability of a closed point lying in a sphere with diameter P_iP_{i+1}.

The edges from level k to $k+1$ are independent from distinct starting points, so the whole process dominates an oriented site percolation with sites open independently with probability $p=1-2\lceil 4r_d/\eta\rceil(\exp(-2^d\eta^d\rho c)+\eta^d)$. It is known [**?**] that there exists a critical probability $p_c<0.7491$ for this process, above which the origin lies in an infinite cluster with positive probability. However, this implies that $P_{0,0}$ is in an infinite cluster of open Voronoi cells with positive probability. Since the set of configurations with an infinite open cluster is shift-invariant, the Ergodic theorem implies that there is almost surely some infinite open cluster. It therefore remains to find values of ρ for which $p>p_c$.

Define η by $\eta^d=(1-p_c)/(32r_d)<1$. Then for large d, $2\lceil 4r_d/\eta\rceil<16r_d$ and $p>p_c$ provided $\exp(-2^d\eta^d\rho c)<\eta^d$. This holds provided

$$\rho > 2^{-d}(1-p_c)^{-1}c^{-1}32r_d\log((32r_d)/(1-p_c)).$$

Since $r_d \sim \sqrt{d/2\pi e}$ this holds provided

$$\rho > C\,2^{-d}\sqrt{d}\,\log d$$

for some constant $C > 0$ independent of d. $\square$

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