Ramsey Unsaturated and Saturated Graphs

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Abstract

A graph is *Ramsey unsaturated* if there exists a proper supergraph of the same order with the same Ramsey number, and *Ramsey saturated* otherwise. We present some conjectures and results concerning both Ramsey saturated and unsaturated graphs. In particular, we show that cycles C_n and paths P_n on n vertices are Ramsey unsaturated for all $n \geq 5$.

1 Results and Conjectures

Throughout this article, $r(G,H)$ will denote the Ramsey number of a pair of graphs (G,H), i.e., the minimum n such that in any coloring of the edges of K_n with colors red and blue, we either obtain a red subgraph isomorphic to G, or a blue subgraph isomorphic to H. When $G=H$ the notation is reduced to $r(G)$.

For years there has been interest in how the Ramsey number of the complete graph grows as its order increases. A favorite question of Erdős (personal communication) was whether one can show that the difference between two consecutive Ramsey numbers is at least quadratic, i.e., whether $r(K_{n+1})-r(K_n) \geq cn^2$ for some positive constant c. Since Kim [9] has shown that there is a constant c_1 such that $r(K_n,K_3) \geq \frac{c_1 n^2}{\log n}$ it is easy to show $r(K_{n+2}) - r(K_n) \geq \frac{c_1 n^2}{\log n}$. Concerning the question of Erdős, the best that is presently known is that the difference of consecutive Ramsey numbers is at least linear [2].

In this paper another growth question is addressed but of a different nature. Given a graph G, is there a nontrivial supergraph of the same order with the same Ramsey number? Also

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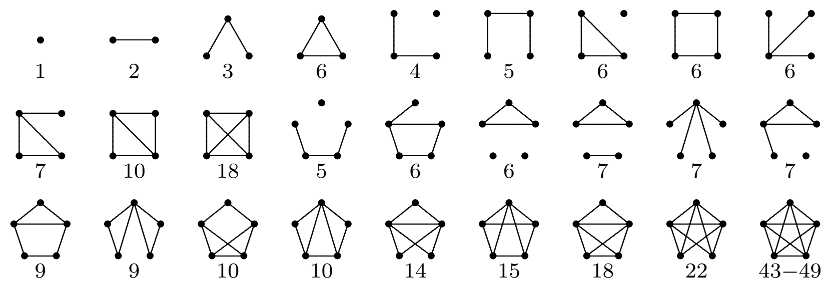


Figure 1: All Ramsey saturated graphs with ≤ 5 vertices and their Ramsey numbers.

there is interest in such edge maximal graphs; graphs in which each nontrivial supergraph of the same order has a larger Ramsey number than the graph itself. This motivates the following definitions.

Definition 1. *A graph G on n vertices is said to be* Ramsey unsaturated *if there exists an edge $e \in E(\overline{G})$ such that $r(G+e) = r(G)$. The graph G is* Ramsey saturated *if $r(G+e) > r(G)$ for all $e \in E(\overline{G})$, i.e., if G is not unsaturated.*

It is well known [11] that $r(K_n) - r(K_n - e) > 0$, e an edge of K_n, for $3 \leq n \leq 6$, with this difference increasing on this interval. Strangely enough, it is not known whether $r(K_n) - r(K_n - e) > 0$ for $n \geq 7$. Surely this must be the case, namely, that $K_n - e$ is Ramsey saturated for all $n \geq 3$. This leads to an even stronger conjecture.

Conjecture 1. *For any $C > 0$ there exists $n_0 = n_0(C)$ such that if $n \geq n_0$ and H is a collection of edges from K_n with $|H| \leq C$, then $K_n - H$ is Ramsey saturated. In particular, $K_n - e$ is Ramsey saturated for large n.*

It is thought that most graphs are Ramsey unsaturated, since most graphs have around one half of the number of possible edges, leaving many choices for an additional edge, one of which could leave the Ramsey number unchanged. This then gives a second conjecture.

Conjecture 2. *Almost all graphs are Ramsey unsaturated.*

For small n we note that 2 out of the 4 graphs on three vertices, 8 out of the 11 graphs on four vertices, and 15 out of the 34 graphs on five vertices are saturated (see Figure 1 and [8]).

The first place to look for unsaturated graphs is in a family of graphs that have few (at most a linear number) of edges, and for which the Ramsey numbers are known. One such family is the collection of cycles and paths.

Cycles are particularly interesting. Given a cycle C_n on n vertices call xy a k-chord if the distance between x and y on C_n is k. Consider first a fixed k with $1 < k < \frac{n}{2}$ and k

relatively prime to n. Note the k-chords of C_n form another cycle of order n on the vertices of C_n. Let $C_n(k)$ be the cycle formed from the k-chords of C_n. One wishes to check whether $r(C_n+k\text{-chord}) = r(C_n)$. Assume $r(C_n+k\text{-chord}) > r(C_n)$ and give $K_{r(C_n)}$ a red-blue edge coloring containing no monochromatic $C_n + k$-chord. Without loss of generality assume this coloring contains a red C_n. Now each k-chord of this red $C_n = C_n(1)$ must be blue, i.e., $C_n(k)$ is blue. Likewise $C_n(k^i)$ is red for all even i, and blue for all odd i (we take k^i modulo n and identify $(n-k)$-chords with k-chords). It is therefore apparent that $k^i \not\equiv \pm 1 \bmod n$ for all odd i. Otherwise, $C_n(k^i) = C_n(1)$ would need to both red and blue. This gives the following theorem.

Theorem 1. *Let n and k be integers with $n \geq 5$, $1 < k < \frac{n}{2}$, and $\gcd(k, n) = 1$. If there is an odd $i > 0$ such that $k^i \equiv \pm 1 \bmod n$ then $r(C_n + k\text{-chord}) = r(C_n)$.* □

Using Theorem 1 we can now show that C_n is Ramsey unsaturated for most values of n. Let $\mathbb{Z}_n^\times$ denote the multiplicative group of integers mod n that are relatively prime to n. Then $|\mathbb{Z}_n^\times| = \phi(n)$, the Euler phi function. If $\phi(n)$ is not a power of 2, it must be divisible by an odd prime p. However, in this case, $\mathbb{Z}_n^\times$ must contain an element $k \not\equiv \pm 1$ of odd order (consider any nontrivial element of a p-Sylow subgroup of $\mathbb{Z}_n^\times$). Considering either k or $(n-k)$-chords and applying Theorem 1 gives the following.

Theorem 2. *If $n \geq 5$ and $\phi(n)$ is not a power of 2, then C_n is Ramsey unsaturated.* □

If $\phi(n)$ is a power of 2, then n must be of the form $n = 2^r p_1 \ldots p_s$ where $p_i = 2^{2^{t_i}} + 1$ are distinct Fermat primes. The only known Fermat primes are 3, 5, 17, 257, and 65537, so there are very few exceptional values of n for which we cannot apply Theorem 2.

The reader should observe that the arguments just given showing most cycles are unsaturated have only used the existence of a monochromatic cycle together with elementary number theoretic arguments on the order of the cycle. One can prove even more using similar techniques when the length of the chord added to the cycle is specified. The following theorem will be proved in the next section.

Theorem 3. *C_n is Ramsey unsaturated for all $n \geq 5$, while C_4 is Ramsey saturated.*

More is expected to be true as is indicated by the following conjecture.

Conjecture 3. *$r(C_n + k\text{-chord}) = r(C_n)$ provided n and k are not both even.*

One could say, if this conjecture is true, that for odd n, C_n is *strongly Ramsey unsaturated*, that is, the Ramsey number is unchanged when adding *any* edge from the complement. The results of Section 2 show that Conjecture 3 is true when n is a prime of the form $4q + 3$.

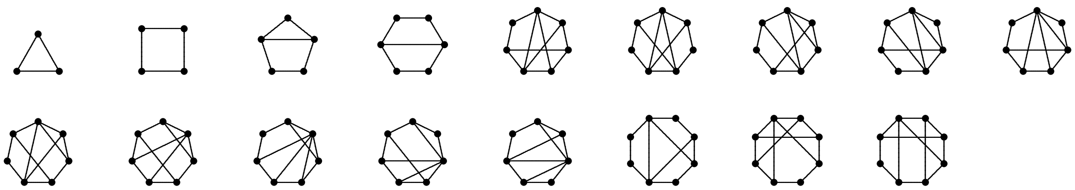


Figure 2: All Ramsey saturated supergraphs G of C_n with $r(G) = r(C_n)$ for $n \leq 8$.

It should be noted that something much stronger than Theorem 3 should be true for large order cycles. One would expect to be able to add more than one edge to the cycle without increasing the Ramsey number. This has been checked by computer for C_7 and C_8. For C_7, any single chord can be added, and up to four specific chords can be added leaving the Ramsey number unchanged. For C_8 any combination of three 3-chords, and some combinations of four 3-chords can be added. It would be nice to be able to show that any constant number of edges can be added to a sufficiently large cycle without increasing its Ramsey number.

Knowing the Ramsey number for even cycles and for paths makes it easy to show paths of order at least 5 are Ramsey unsaturated. Since $r(P_{2n}) = r(C_{2n}) = 3n - 1$ for $n \geq 3$ (see [6] and [5], [12]), it follows that $P_{2n} = (a_1, \ldots, a_{2n})$ together with an edge $a_1 a_{2n}$ appears monochromatically in any red-blue coloring of K_{3n-1}. Hence paths of even order at least 6 are unsaturated. Since $r(P_{2n+1}) = 3n \geq r(C_{2n})$ for $n \geq 2$, one may assume the red-blue edge colored K_{3n} contains a red C_{2n}. But either there is a red edge from C_{2n} to the remaining n vertices of K_{3n} or there is a blue $K_{2n,n}$. In either case, there is a monochromatic graph consisting of $P_{2n+1} = (a_1, \ldots, a_{2n+1})$ with chord $a_1 a_{2n}$. Hence paths of odd order at least 5 are also unsaturated, giving the following result.

Theorem 4. *Paths of order $n \geq 5$ are Ramsey unsaturated.* □

Note that P_n is Ramsey saturated for $n = 2$, 3, and 4 (see Figure 1). Having considered paths, what can be said about trees? For stars [3],

$$r(K_{1,n-1}) = \begin{cases} 2n-3 & \text{for } n \text{ odd, } n \geq 3; \\ 2n-2 & \text{for } n \text{ even, } n \geq 2, \end{cases}$$

while it is easy to show $r(K_{1,n-1} + e) = 2n - 1$ for $n \geq 4$. Therefore all stars are saturated. Stars seem to be the exception (as far as trees are concerned). The first test case would be the star $K_{1,n-2}$ with one edge subdivided. Let S_n be the graph formed by joining the path $v_1 v_2 v$ to the central vertex v of the star with endvertices $v_3, \ldots, v_{n-1}$. Further let S_n^e be the graph obtained from S_n by adding the edge $v_1 v_3$. One can show (see Lemma 4) that

$$r(S_n) = r(S_n^e) = \begin{cases} 2n-5 & \text{for } n \text{ even, } n \geq 6; \\ 2n-4 & \text{for } n \text{ odd, } n \geq 5, \end{cases}$$

so that S_n is unsaturated. Therefore one might expect other non-star trees to be unsaturated.

Conjecture 4. *Every non-star tree of order $n \geq 5$ is Ramsey unsaturated.*

It should be noted that if an edge e is added to a tree T_n on n vertices creating an odd cycle then $r(T_n + e) \geq 2n - 1$. This follows from the red-blue edge coloring of K_{2n-2} in which one color class is $K_{n-1,n-1}$. Further it was recently announced that M. Ajtai, T. Komlós, and E. Szemerédi have proved (for n large) the Erdös-Sós conjecture [4] that every graph of average degree larger than $n - 2$ contains all trees on n vertices. From this it easily follows (look at the dominant color) that for large n the Ramsey number for any tree T_n satisfies $r(T_n) \leq 2n - 2$. Hence Conjecture 4 can be strengthened.

Conjecture 5. *If T_n is a non-star tree of order $n \geq 5$, then $r(T_n + e) = r(T_n)$ for each edge e such that $T_n + e$ is bipartite.*

There are graphs with many edges that are Ramsey unsaturated. It is known that if a graph G is formed from K_n by adding a new vertex and joining it to precisely two vertices of K_n then $r(G) = r(K_n)$ for n large (see [2]). Hence the graph H formed by joining a new vertex x to one vertex of K_n makes H a Ramsey unsaturated graph. This is a graph with many edges. Indeed, it misses only a linear number of edges.

Conjecture 6. *The graph formed by joining a new vertex to a constant number of vertices of K_n, with n large, is unsaturated.*

The next objective is to find a large collection of Ramsey saturated graphs on n vertices. Earlier it was noted that

$$r(S_n^e) = \begin{cases} 2n - 5 & \text{for } n \text{ even, } n \geq 6; \\ 2n - 4 & \text{for } n \text{ odd, } n \geq 5. \end{cases}$$

This graph is bipartite with parts of order 2 and $n - 2$. To obtain a saturated graph one may add edges from $\overline{S_n^e}$ keeping the graph bipartite as long as possible so that the Ramsey number of the resulting graph is the same as $r(S_n^e)$. When this is no longer possible a Ramsey saturated graph is obtained because if any further edge is added then an odd cycle emerges, thus increasing the Ramsey number to at least $2n - 1$. Starting with different subdivided stars of order n, the same procedure yields distinct Ramsey saturated graphs as shown in the next theorem.

Theorem 5. *For each $n \geq 4$ there exists at least $\lfloor \frac{n-2}{2} \rfloor$ non-isomorphic bipartite Ramsey saturated graphs.*

Proof. For each fixed $c \geq 1$ and $n \geq 2c+2$, subdivide c edges of a $K_{1,n-1-c}$ yielding a bipartite graph G with parts of size $c+1$ and $n-c-1 \geq c+1$.

To see that $r(G) \leq 2n-3$, give K_{2n-3} a red-blue edge coloring. Let x be an arbitrary vertex of this colored graph and assume without loss of generality that x has a set A of red neighbors with $|A| = n-2$. Set $B = V(K_{2n-3} \setminus (A \cup \{x\}))$. If there is a red c-matching from B to A then the colored K_{2n-3} has a red copy of G. Suppose therefore that there is no such red c-matching. By König, there is a subset $S \subseteq A$ of vertices who between them have at most $k = |S| - (n-2) + (c-1)$ red neighbors in B. Thus there is a blue $K_{|S|,n-2-k}$. But $k \geq 0$ so $|S| \geq n-c-1 \geq c+1$, and $k \leq c-1$, so $n-2-k \geq n-1-c$. Thus this blue $K_{|S|,n-2-k}$ contains a blue $K_{c+1,n-1-c}$ and hence a blue copy of G.

Therefore, since any supergraph of G of order n containing an odd cycle has Ramsey number at least $2n-1$, a bipartite Ramsey saturated graph will be obtained from G by adding bipartite edges to G until any addition increases the Ramsey number. Since each such graph is distinct for distinct values of c, this procedure will yield $\lfloor \frac{n-2}{2} \rfloor$ distinct non-isomorphic bipartite Ramsey saturated graphs. It is worth noting that by probabilistic arguments, $r(K_{2,n-2}) > (4-\epsilon)n$, so that for large n

$$2n-3 < r(K_{2,n-c-1}) \leq r(K_{c+1,n-c-1}).$$

Hence these Ramsey saturated graphs are strictly smaller than $K_{c+1,n-c-1}$, the maximal bipartite supergraph of G □

There should be a much larger family of Ramsey saturated graphs on n vertices than that given in Theorem 5.

Conjecture 7. *There exist constants $c > 0$ and $\epsilon > 0$ such that at least $cn^{1+\epsilon}$ of the n-vertex graphs are Ramsey saturated.*

A book B_n is a set of n triangles with a common edge, being otherwise disjoint. Also if one adds an edge to a book it changes from a 3-chromatic graph to one that is 4-chromatic. Since adding an edge to a star changes a 2-chromatic graph to one which is 3-chromatic, and since that star is Ramsey saturated, it is natural to wonder if a book is also Ramsey saturated. In [10] it is shown that this is far from being the case, namely, as many as cn^2 appropriately selected edges may be added to B_n without increasing the Ramsey number.

2 Proofs

In order to prove Theorem 3 first a lemma is needed.

Lemma 1. *If $n > 6$, $n \neq 8, 10, 12, 24$, and if k is relatively prime to n with $k^2 \not\equiv \pm 1 \bmod n$ then $r(C_n + k\text{-chord}) = r(C_n)$.*

Proof. Suppose the result is false giving $K_{r(C_n)}$ a red-blue coloring with no monochromatic $C_n + k$-chord. We may assume this coloring contains a red C_n. Since k is relatively prime to n and C_n has no red k-chord, the set of k-chords $C_n(k)$ forms a blue C_n. Likewise the set of k-chords of this blue $C_n(k)$ again forms a red $C_n = C_n(k^2)$, different from the original red C_n since $k^2 \not\equiv \pm 1 \bmod n$.

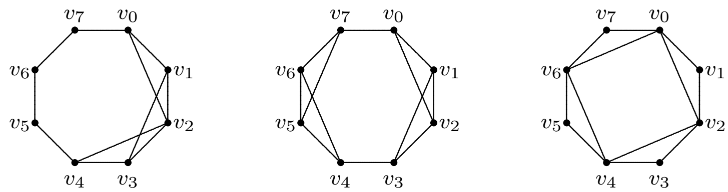
Set $y \equiv k^2 \bmod n$ and note that the above argument shows that *any* monochromatic C_n has all its y-chords of the same color. The next objective is to show that a monochromatic C_n (say a red one) has all odd length chords red as well. This is done inductively. Let the red $C_n = (a_0, a_1, \ldots, a_{n-1})$ and consider the two consecutive red chords $a_i a_{i+y}$, $a_{i+1} a_{i+y+1}$, (all indices taken mod n). But then the 3-chord $a_{i-1} a_{i+2}$ is red since it is a y-chord of the red $C_n = (a_{i+1}, a_{i+2}, \ldots, a_{i+y}, a_i, a_{i-1}, \ldots, a_{i+y+1})$. Applying this argument for each i shows that all 3-chords of C_n are red. Now assume all $(2j-1)$-chords are red in each red C_n. Consider the two consecutive red $(2j-1)$-chords $a_i a_{i+2j-1}$, $a_{i+1} a_{i+2j}$ in the red $C_n = (a_0, \ldots, a_{n-1})$. But then the $(2j+1)$-chord $a_{i-2} a_{i+2j-1}$ is red since it is a 3-chord in the red $C_n = (a_{i+2j-1}, a_i, a_{i-1}, \ldots, a_{i+2j}, a_{i+1}, \ldots, a_{i+2j-2})$. Therefore all $(2j+1)$-chords of any red C_n are red. Thus all odd chords are red in any red C_n.

But all k-chords are blue, so k is even. But then, since k is relatively prime to n, n must be odd, implying that $n-k$ is odd. Hence all $(n-k)$-chords are red, and since every k-chord is an $(n-k)$-chord, a final contradiction is reached, completing the proof of the lemma.

It is easy to see that the only values of n where $k^2 \equiv \pm 1 \bmod n$ for all k relatively prime to n, $n > 6$, are $n = 8$, 10, 12, and 24. □

We note that if $n > 3$ is a prime of the form $4q + 3$ then k is relatively prime to n and $k^2 \not\equiv \pm 1 \bmod n$ for all k with $1 < k < \frac{n}{2}$. Thus *any* single chord can be added to C_n without increasing the Ramsey number.

Proof of Theorem 3. Now $r(C_4) = 6$ while $r(C_4 + 2\text{-chord}) = r(K_4 - e) = 10$, so C_4 is Ramsey saturated. Hence Theorem 3 will follow provided we can show that $r(C_n + k\text{-chord}) = r(C_n)$ for some value of k in each of the cases $n = 5$, 6, 8, 10, 12, and 24. In each case we checked the result by computer. For $n = 24$ we found that all red-blue colorings of $K_{12,12}$ that contains a red C_{24} also contain both monochromatic $C_{24}+5$-chords and $C_{24}+7$-chords. Hence, there is certainly no coloring of $K_{r(C_{24})}$ that avoids these graphs. Note that as in Lemma 1, one can construct a monochromatic $C_n + k$-chord using just the vertices of a monochromatic C_n. Unfortunately, this does not hold for the other values of n. For $n = 8$, 10, and 12, one can check using a computer that any red-blue coloring of K_{n+1} that contains a monochromatic C_n also contains a monochromatic $C_n + k$-chord, where $k = 3$, 3, and 5 respectively. In particular, only one more vertex in addition to the vertices of the cycle needs to be considered. For $n = 5$ and 6, more than one additional vertex is required in order to obtain a monochromatic $C_n + k$-chord. In [7] Hendry has shown that $r(C_6 + 3\text{-chord}) = r(C_6) = 8$, and in [8] that $r(C_5 + 2\text{-chord}) = r(C_5) = 9$.



Three 2-chords   Type I   Type II

Figure 3: $C_8 + 3$-chords

Although we do not have explicit proofs for $n = 12$ and $n = 24$, we shall give direct proofs for the cases $n = 8$ (Lemma 2) and $n = 10$ (Lemma 3). □

Lemma 2.

$$r(C_8 + \textit{3-chord}) = r(C_8) = 11 \ .$$

Proof. Color the edges of K_{11} red-blue. Then since $r(C_8) = 11$ (see [5] and [12]), we may assume there is a red $C_8(R) = (v_0, v_1, \ldots, v_7)$. We also assume that there is no monochromatic $C_8 + 3$-chord on these eight vertices, so in particular, the 3-chords of $C_8(R)$ form a blue cycle $C_8(B) = (v_0, v_3, v_6, v_1, v_4, v_7, v_2, v_5)$.

First we observe that no three red 2-chords are consecutive on $C_8(R)$. Assuming on the contrary that v_0v_2, v_1v_3, v_2v_4 are red, $(v_0, v_2, v_1, v_3, \ldots, v_7)$ would be a red C_8 with the red 3-chord v_2v_4. The same is true for the blue 2-chords of $C_8(B)$. Because every 2-chord of $C_8(B)$ is also a 2-chord of $C_8(R)$, we obtain easily that after appropriate renumbering of the vertices, the coloring of the 2-chords is one of the following two types. Either v_0v_2, v_1v_3, v_4v_6, v_5v_7 are red and v_0v_6, v_1v_7, v_2v_4, v_3v_5 are blue (Type I); or (v_0, v_2, v_4, v_6) is a red C_4 and (v_1, v_3, v_5, v_7) is a blue C_4 (Type II). Let $x \notin V(C_8(R))$ be a new vertex. We can assume without loss of generality that x has at least four red adjacencies to $C_8(R)$. If no two consecutive vertices on $C_8(R)$ are red adjacent to x then v_ix, $v_{i+2}x$ and $v_{i+4}x$ are red for some i. Then $(v_i, x, v_{i+2}, v_{i+3}, \ldots, v_{i-1})$ is a red C_8 with the red 3-chord $v_{i+4}x$. Therefore some two consecutive vertices of $C_8(R)$ are red adjacent to x. If these two vertices, say v_i, v_{i+1}, lie in some red 2-chord v_iv_{i+2}, then if v_jv_{j+2} is a red 2-chord disjoint from v_iv_{i+2}, $(v_i, x, v_{i+1}, \ldots, v_j, v_{j+2}, \ldots)$ is a red C_8 with a red 3-chord v_iv_{i+2}. The only remaining case is of Type I with the four red adjacencies of x to $C_8(R)$ being v_3x, v_4x, v_7x, and v_0x. Then $(x, v_0, v_2, \ldots, v_7)$ is a red C_8 with the red 3-chord v_3x. □

Lemma 3.

$$r(C_{10} + 3\textit{-chord}) = r(C_{10}) = 14 \ .$$

Proof. Color the edges of K_{14} red-blue. Then since $r(C_{10}) = 14$ (see [5] and [12]), we may assume there is a red $C_{10}(R) = (v_0, \ldots, v_9)$. We may also assume that there is no

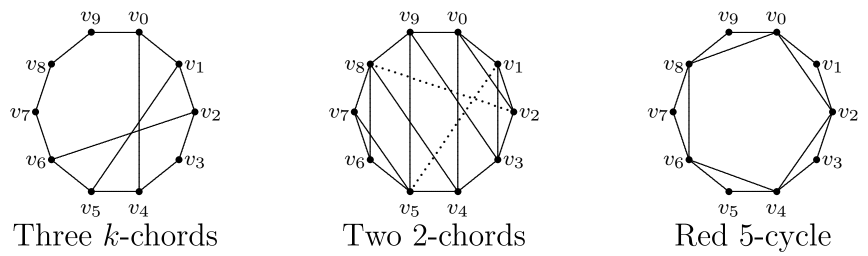


Figure 4: $C_{10} + 3$-chords

monochromatic $C_{10} + 3$-chord on these ten vertices. In particular, the 3-chords of $C_{10}(R)$ form a blue cycle $C_{10}(B) = (v_0, v_3, v_6, v_9, v_2, v_5, v_8, v_1, v_4, v_7)$.

(a) *No three red k-chords are consecutive on $C_{10}(R)$, for $2 \leq k \leq 5$.*

Assume on the contrary that v_0v_k, v_1v_{k+1}, and v_2v_{k+2} are consecutive red k-chords on $C_{10}(R)$. Then $(v_0, v_k, v_{k-1}, \ldots, v_1, v_{k+1}, \ldots, v_9)$ is a red C_{10} with red 3-chord v_2v_{k+2}. Note that the same statement is obviously also true for the blue k-chords of $C_{10}(B)$.

(b) *There are no two red 2-chords consecutive on $C_{10}(R)$.*

Assume on the contrary that v_0v_2 and v_1v_3 are red. In the red cycle $(v_0, v_2, v_1, v_3, \ldots, v_9)$ the chords v_1v_5 and v_2v_8 are 3-chords, hence they are blue. In the blue cycle $(v_6, v_9, v_2, v_8, v_5, v_1, v_4, v_7, v_0, v_3)$ the chords v_6v_8 and v_5v_7 are 3-chords, hence they are red.

Because v_1v_5, v_2v_8, and either of v_5v_9 or v_4v_8 are consecutive 2-chords of $C_{10}(B)$, both v_5v_9 and v_4v_8 are red by (a). Now by symmetry, using the red chords v_5v_7 and v_6v_8, the chords v_0v_4 and v_3v_9 are also red (see Figure 4).

Similarly, looking at the blue $C_{10}(B)$, v_1v_5 and v_2v_8 are consecutive 2-chords, so we get a similar picture in blue chords of $C_{10}(B)$. The chord v_4v_9 appears symmetrically in both $C_{10}(R)$ and $C_{10}(B)$. Hence without loss of generality we may assume v_4v_9 is blue. Then by (a), either v_1v_6 or v_2v_7 is red. Suppose v_1v_6 is red. Then $(v_1, v_2, v_0, v_9, v_3, v_4, v_8, v_7, v_5, v_6)$ is a red C_{10} with red 3-chord v_2v_3.

(c) *There is a red 5-cycle formed from 2-chords of $C_{10}(R)$.*

Observe that three consecutive blue 4-chords of $C_{10}(B)$ correspond to 2-chords of $C_{10}(R)$ of the form $v_{i+1}v_{i+3}$, $v_{i+4}v_{i+6}$, and $v_{i+7}v_{i+9}$. By (a), these are not all blue. In particular, if $v_{i+4}v_{i+6}$ and $v_{i+7}v_{i+9}$ are blue, then both $v_{i+1}v_{i+3}$ and $v_{i+10}v_{i+2}$ are red. This contradicts (b). Therefore v_0v_2 is red implies that v_1v_9 and v_1v_3 are blue, then v_0v_8 and v_2v_4 are red, and so on. In this way one obtains that $(v_0, v_2, v_4, v_6, v_8)$ is a red cycle. Similarly $C_{10}(B)$ has a blue cycle formed from the 2-chords of $C_{10}(B)$.

Without loss of generality, we may assume that a new vertex $x \notin V(C_{10}(R))$ is adjacent to $C_{10}(R)$ with at least five red edges. In the case when v_1x and v_2x are red, $(v_0, v_1, x, v_2, \ldots, v_8)$ is a red C_{10} with red 3-chord v_0v_2. If x has no two consecutive red adjacencies to the cycle $C_{10}(R)$, then $v_i x$, $v_{i+2}x$ and $v_{i+4}x$ are red for some i. In that case $(v_i, x, v_{i+2}, v_{i+3}, \ldots, v_{i-1})$ is a red C_{10} with red 3-chord $v_{i+4}x$. $\square$

Finally, we calculate the Ramsey numbers $r(S_n)$ and $r(S_n^e)$.

Lemma 4.

$$r(S_n) = r(S_n^e) = \begin{cases} 2n-5 & \textit{for } n \textit{ even, } n \geq 6; \\ 2n-4 & \textit{for } n \textit{ odd, } n \geq 5, \end{cases}$$

Proof. First note that $r(S_n^e) \geq r(S_n) \geq r(K_{1,n-2})$ and $r(K_{1,n-2}) = 2n-5$ if $n \geq 6$ is even and $2n-4$ if $n \geq 5$ is odd [3]. Now suppose we have a red-blue coloring on $K_{r(K_{1,n-2})}$. We can assume there exists a red $K_{1,n-2}$, say, so there is a vertex v whose set R of red neighbors satisfies $|R| \geq n-2$. Let B be the set of blue neighbors of v If $u \in B$ has two red neighbors in R, we obtain a red S_n^e. Hence we may assume u has at most one red neighbor in R. But then there are at least $n-2$ blue neighbors of u in $R \cup \{v\}$ and for any two vertices of B, there are at least $n-3 \geq 2$ common neighbors. Hence there is a blue S_n^e if $|B| \geq 2$. On the other hand, if $|B| \leq 1$ then $|R| \geq n - |B| \geq n-1$ (since $(2n-5)-1 \geq n$ for $n \geq 6$ and $(2n-4)-1 = n$ for $n = 5$). If there is a red P_3 in R, we obtain a red S_n^e in $R \cup \{v\}$, thus the red components in $R \cup B$ are isolated edges except possibly for one red P_3 meeting B. In particular we have a blue $K_{2,n-2}$ in $R \cup B$, and hence a blue S_n^e. $\square$

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