# Vertex-distinguishing edge colorings of random graphs

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**Abstract**

A proper edge coloring of a simple graph $G$ is called *vertex-distinguishing* if no two distinct vertices are incident to the same set of colors. We prove that the minimum number of colors required for a vertex-distinguishing coloring of a random graph of order $n$ is almost always equal to the maximum degree $\Delta(G)$ of the graph.

## 1 Introduction

Let $G$ be a simple graph with $n$ vertices. For $d \geq 0$ write $n_d$ for the number of vertices in $G$ of degree $d$. Let $\chi'(G)$ be the minimum number of colors required in a proper edge coloring of $G$. If we have such a proper coloring with colors $\{1, \ldots, k\}$ and $v$ is a vertex of $G$, denote by $S(v)$ the set of colors used to color the edges incident to $v$.

A proper edge coloring of a graph is said to be *vertex-distinguishing* if each pair of vertices is incident to a different set of colors. In other words, $S(u) \neq S(v)$ for all $u \neq v$. A vertex-distinguishing proper edge coloring will also be called a *strong* coloring. A graph has a strong coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec*-graph. The minimum number of colors required for a strong coloring of a vdec-graph $G$ will be denoted $\chi'_s(G)$. If $G$ is not a vdec-graph then we write $\chi'_s(G) = \infty$.

The concept of vertex-distinguishing colorings was introduced independently by Aigner, Triesch, and Tuza, by Hořňák and Soták, and by Burris and Schelp, and has been considered in several papers [1, 2, 3, 4, 5, 8, 9, 11, 12]. In [8] Burris and Schelp made the following conjecture:

**Conjecture 1** *Let $G$ be a vdec-graph and let $k = k(G)$ be the minimum integer such that $\binom{k}{d} \geq n_d$ for all $d$ with $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) = k$ or $k + 1$.*

Conjecture 1 was strengthened to give conjectural criteria for when $\chi'_s(G) = k$ and when $\chi'_s(G) = k + 1$ in [4]. In practice this strengthened conjecture suggests that the "usual" value is $k$, with $k+1$ only occurring when some parity constraint forces it. This would be analogous to the normal edge chromatic number $\chi'(G)$ which by Vizing's Theorem is either $\Delta$ or $\Delta + 1$ with the "usual" value being $\Delta$ (see [10]).

The strengthened conjecture (and hence the exact value of $\chi'_s(G)$) is known for complete graphs, complete bipartite graphs, and many trees [8]. More recently it has been proved for unions of cycles [2], unions of paths [2], and for graphs of small order [4].

Conjecture 1 is proved for graphs of large maximum degree in [3], where the following result is also given.

**Theorem 1** *Assume $k \geq \chi'(G)$. If $n_0, n_1, n_2 \leq 1$, $n_3, n_4, n_k \leq 2$, $n_{k-1} \leq k+1$, and for $5 \leq d \leq k-2$,*

$$n_d \leq \frac{d-4}{d-3} \min\left\{2\binom{k-3}{d-3}, \binom{k}{d}\right\} - 2, \tag{1}$$

*then we can find a strong coloring of $G$ with at most $k+1$ colors.*

Let $G_{n,p}$ be a random graph on $n$ vertices with edge probability $p = p(n)$. If $\frac{pn}{\log n}, \frac{(1-p)n}{\log n} \to \infty$ then almost all such graphs satisfy the conditions of Theorem 1 with $k = \Delta(G)$, so $\chi'_s(G) \leq \Delta(G)+1$. Since it is clear that $\chi'_s(G) \geq k(G) \geq \Delta(G)$, we know Conjecture 1 holds for these graphs. In this paper we will prove the stronger conjecture holds almost always by showing that for almost all graphs $\chi'_s(G) = \Delta(G)$. This is analogous to (and implies) the main result of [10], that for almost all graphs $\chi'(G) = \Delta(G)$.

For $v \in V(G)$, define a *split* at $v$ to be a new graph $G'$ in which the vertex $v$ has been replaced by two non-adjacent vertices $v_1$ and $v_2$ with the neighborhood of $v$ in $G$ equal to the disjoint union of the neighborhoods of $v_1$ and $v_2$ in $G'$. We call a split an *$r$-split* if the degree of $v_1$, say, is $r$. In Section 2 we shall prove:

**Theorem 2** *Let $G$ be a graph with precisely one vertex, $v$ say, of maximum degree and let $k \geq \Delta(G)$. If there exists a 2-split $G'$ of $G$ at $v$ with $\chi'_s(G') \leq k-1$ then $\chi'_s(G) \leq k$.*

Theorems 1 and 2 have the following consequence.

**Corollary 3** *Write $\Delta = \Delta(G)$. If $n_\Delta = 1$, $n_2, n_{\Delta-1} = 0$, $n_0, n_1, n_{\Delta-2} \leq 1$, $n_3, n_4 \leq 2$, $n_{\Delta-3} \leq \Delta - 1$ and for $5 \leq d \leq \Delta - 4$,*

$$n_d \leq \frac{d-4}{d-3} \min\left\{2\binom{\Delta-5}{d-3}, \binom{\Delta-2}{d}\right\} - 2, \tag{2}$$

*then $\chi'_s(G) = \Delta$.*

*Proof.* Let $v$ be the (unique) vertex of degree $\Delta$ in $G$. The conditions imply that $\Delta \geq 8$ and that any 2-split $G'$ of $G$ at $v$ satisfies the degree sequence conditions of Theorem 1 with $k = \Delta - 2$. The strengthened version of Vizing's theorem proved in [14] states that if $\chi'(G') > \Delta(G')$ then $G'$ contains at least three vertices of maximum degree. However there are at most two vertices of maximum degree $\Delta(G') = k$ in $G'$, so $k \geq \chi'(G')$. Hence $\chi'_s(G') \leq \Delta - 1$ and the result now follows from Theorem 2 with $k = \Delta$. □

The following theorem will be proved in Section 3 by showing that almost all graphs satisfy the conditions of Corollary 3.

**Theorem 4** *If $G = G_{n,p}$ is a random graph on $n$ vertices with edge probability $p = p(n)$ and if $\frac{pn}{\log n}, \frac{(1-p)n}{\log n} \to \infty$ as $n \to \infty$ then $\mathbb{P}(\chi'_s(G) = \Delta) \to 1$ as $n \to \infty$.*

## 2 The proof of Theorem 2

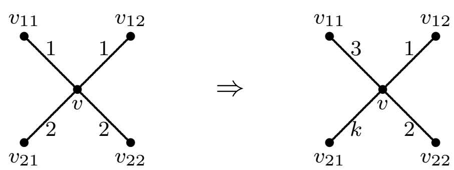
Fix a 2-split $G'$ and a strong coloring of $G'$ with $k-1$ colors. Now $S(v_1)$ is a set of two colors and $S(v_2)$ is a set of $\Delta - 2 \leq k-2$ colors. Identifying the vertices $v_1$ and $v_2$ gives a coloring of $G$ which is vertex-distinguishing and proper except possibly at $v$, where at most two colors are incident to $v$ twice. We can distinguish three cases.

Case I: No color is repeated at $v$. In this case, the coloring is proper and we are done. The vertex $v$ is distinguished from all the others since it is the only vertex of degree $\Delta$.

Case II: One color is repeated at $v$, say color 1. Re-color one of the two edges from $v$ that are colored 1, $vw$ say, with the (so far unused) color $k$. The result is a strong coloring with $k$ colors since $S(v)$ is the only set with size $\Delta$, $S(w)$ is the only other set containing $k$, and all the other sets $S(u)$, $u \neq v, w$, are unaltered.

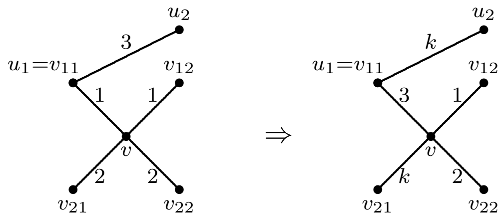
Case III: Two colors are repeated at $v$, say colors 1 and 2. The color $k$ is unused anywhere and since $\Delta \leq k$, there exists some other color, say 3, which is not incident to $v$. Let $v_{ij}$, $1 \leq i, j \leq 2$, be the four distinct vertices with $vv_{ij}$ colored with $i$. For any two colors $a$ and $b$ and set of colors $S$ define $i_{ab}S$ to be the set obtained by replacing any occurrence of $a$ by $b$ and any occurrence of $b$ by $a$ in $S$.

First we shall attempt to change the color of one of the edges $vv_{ij}$, say $vv_{11}$, to 3. This may cause one of two problems. The first is that the new coloring may fail to be proper at $u_1 = v_{11}$ since $v_{11}$ may already be incident to color 3. The other problem is that the vertex $v_{11}$ may now have the same color set as some other vertex $u_1$. In either case there is a vertex $u_1$ with an edge $u_1u_2$ colored 3 and $S(v_{11}) = i_{13}S(u_1)$. (Here and below $S(x)$ will refer to the set of colors meeting $x$ in the original coloring.) Note that $u_1, u_2 \neq v$ since $3 \in S(u_1), S(u_2)$. If no such $u_1$ exists then we can re-color $vv_{11}$ with 3 and we are reduced to case II where we can re-color, e.g., $vv_{21}$ with $k$ to get a strong coloring.



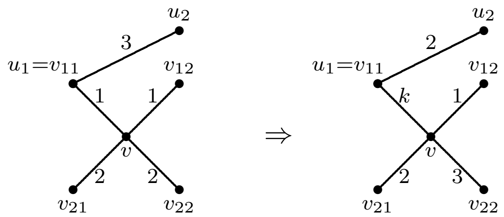
Assuming no $u_1$ with $S(v_{11}) = i_{13}S(u_1)$.

Assume now that at least one of the sets $S(v_{ij})$ contains color 3. Without loss of generality we may assume $3 \in S(v_{11})$, so $u_1 = v_{11}$. If we re-color $vv_{11}$ with 3 and re-color $vv_{21}$ and $u_1u_2$ with $k$, then the only color sets that have changed are at $v$, $u_1$, $u_2$, and $v_{21}$, each of which now meets color $k$ and is thus distinguished from all other vertices. The vertex $u_2$ now does not meet color 3 so is distinguished from $u_1$. As before, $v$ is distinguished from all other vertices since it is the only vertex of degree $\Delta$. Hence the coloring will be strong unless $v_{21}$ is not distinguished from either $u_1$ or $u_2$. In other words, the coloring will be strong unless $S(v_{21}) = i_{21}S(u_1)$ or $S(v_{21}) = i_{23}S(u_2)$. (The second possibility also covers the case when $v_{21} = u_2$.)



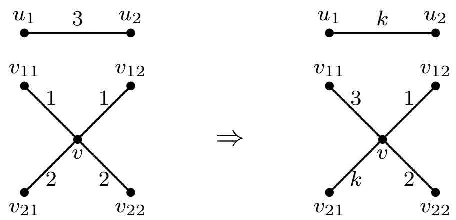

Assuming $S(v_{21}) \neq i_{21}S(u_1), i_{23}S(u_2)$.

By symmetry, we are also done if $S(v_{22}) \neq i_{21}S(u_1), i_{23}S(u_2)$. Since $S(v_{21}) \neq S(v_{22})$, we can now assume without loss of generality that $S(v_{21}) = i_{21}S(u_1)$ and $S(v_{22}) = i_{23}S(u_2)$. In this case, re-color the vertices as shown below to give a strong coloring. The color sets $S(u_2)$ and $S(v_{22})$ are swapped (or unchanged if $u_2 = v_{22}$) and all other color sets are unchanged except at $v_{11}$ and $v$ which see distinct color sets and are the only color sets containing $k$. The coloring is proper at $u_1$ since $S(u_1) = i_{21}S(v_{21})$ and $v_{21} \neq u_1$ which imply $2 \notin S(u_1)$.




Assuming $S(v_{21}) = i_{21}S(u_1)$, $S(v_{22}) = i_{23}S(u_2)$.

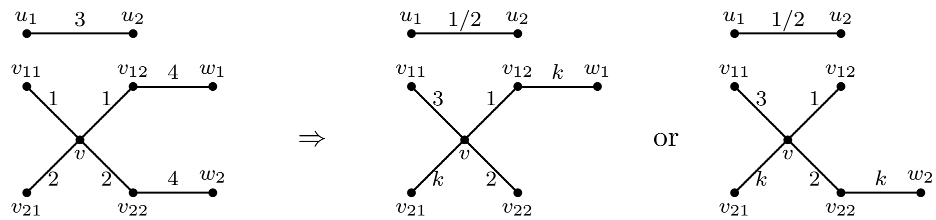
Now assume $3 \notin S(v_{ij})$ for all $1 \leq i, j \leq 2$. Hence $u_1$ and $u_2$ are distinct from all the $v_{ij}$ and $v$. Once again, color $vv_{11}$ with 3 and both $u_1u_2$ and $vv_{21}$ with $k$. Now $v_{11}$ is now properly colored, does not meet color $k$ and is distinguished from all other vertices. The color sets of $u_1$ and $u_2$ are both changed by $i_{3k}$, so $u_1$ and $u_2$ remain properly colored and distinguished from each other. Hence the coloring will be strong unless $v_{21}$ is no longer distinguished from $u_1$ or $u_2$. In other words, unless $S(v_{21}) = i_{23}S(u_1)$ or $i_{23}S(u_2)$.




Assuming $S(v_{21}) \neq i_{23}S(u_1), i_{23}S(u_2)$.

Re-coloring $vv_{22}$ instead of $vv_{21}$ we are also done if $S(v_{22}) \neq i_{23}S(u_1), i_{23}S(u_2)$. Hence by symmetry we can assume $S(v_{21}) = i_{23}S(u_1)$ and $S(v_{22}) = i_{23}S(u_2)$. Thus $S(v_{ij}) = i_{i3}S(u_j)$ for all $i, j$ except possibly $(i, j) = (1, 2)$. Now apply the same argument to $v_{22}$ in place of $v_{11}$ with colors 1 and 2 interchanged. The vertex $u_2$ is the "$u_1$" for $v_{22}$ and the edge $u_2u_1$ takes the place of $u_1u_2$. Hence we can assume there exists a single edge $u_1u_2$ with $S(v_{ij}) = i_{i3}S(u_j)$ for all $1 \leq i, j \leq 2$ and the $u_k$ are distinct from all $v_{ij}$.

Since $S(u_1) \neq S(u_2)$, we may assume by symmetry that there is a color, 4 say, which is in $S(u_2)$ but not in $S(u_1)$. (We know $S(u_i) \cap \{1,2,3,k\} = \{3\}$ for $i = 1,2$ so $S(u_1)$ and $S(u_2)$ must differ in some other color.) Hence $4 \in S(v_{12}), S(v_{22})$. The edge $u_1u_2$ is not incident to either color 1 or 2. We shall re-color $u_1u_2$ with either 1 or 2, $vv_{11}$ with 3 and $vv_{21}$ with $k$. The only remaining problem is that $u_2$ is not distinguished from one of $v_{12}$ or $v_{22}$. Let $v_{i2}w_i$ be the edges colored with 4 and re-color $v_{12}w_1$, say, with $k$. Note that $w_1 \neq v_{11}, v_{21}, u_1$ since these vertices do not meet color 4. It is however possible that $w_1 = u_2$ or $w_1 = v_{22}$. If $w_1 = u_2$, color $u_1u_2$ with 2, otherwise color it 1. Apart from at $v$, $w_1$, $v_{12}$, and $v_{21}$, all color sets are unchanged or permuted. Vertex $w_1$ is distinguished from $v_{12}$ since when $w_1 \neq u_2$, the only color change either see is on the edge $v_{12}w_1$ and when $w_1 = u_2$ only $v_{12}$ sees color 1. The vertex $v_{12}$ is distinguished from $v_{21}$ since only $v_{12}$ meets color 1. Hence this re-coloring succeeds in giving a strong coloring of $G$ unless $w_1$ is not distinguished from $v_{21}$. This occurs only when $w_1 \neq u_2, v_{22}$ (since $v_{21}$ does not meet color 2) and $i_{4k}S(w_1) = i_{2k}S(v_{21}) = i_{3k}S(u_1)$. If $i_{4k}S(w_1) = i_{3k}S(u_1)$ then color $v_{22}w_2$ with $k$ instead. Now $w_1 \neq w_2$ since otherwise it would meet color 4 twice in the original coloring. Hence $S(w_1) \neq S(w_2)$, $i_{4k}S(w_2) \neq i_{3k}S(u_1)$ and we are done. □




Assuming $u_j$ and $v_{ij}$ all distinct and $S(v_{ij}) = i_{i3}S(u_j)$

## 3 Proof of Theorem 4.

It is sufficient to show that for almost all graphs the conditions of Corollary 3 hold. The details are somewhat tedious so we shall only sketch them here. The ideas used here follow the proofs in [6, 7, 10, 13].

Let $0 < \epsilon < 0.2$ be some fixed small number and let $\mathcal{D} = \{d \in \mathbb{Z} : |\frac{d}{n-1} - p| \leq \epsilon pq\}$ where $q = 1-p$. Write $d_v$ for the degree of $v \in V(G)$. We shall show that it is enough that there exists an integer $k_0 \geq 7$ for which each of the following events occurs with small probability.

$E_1$ $\quad \forall v : d_v \in \mathcal{D}$ and $d_v < k_0 + 3$,

$E_2$ $\quad \exists u, v : d_u, d_v \geq k_0,\ d_u, d_v \in \mathcal{D}$ and $d_u \leq d_v \leq d_u + 1$,

$E_3$ $\quad \exists d \leq k_0 : d \in \mathcal{D}$ and $n_d > 2^{k_0-1-d}$,

$E_4$ $\quad \exists d \notin \mathcal{D} : n_d > 0$.

Assume events $E_1$–$E_4$ fail. Since $E_1$ and $E_4$ fail, $\Delta \in \mathcal{D}$ and $\Delta \geq k_0 + 3$. Since $E_2$ also fails, $n_\Delta = 1$, $n_{\Delta-1} = 0$ and $n_d \leq 1$ for $k_0 \leq d \leq \Delta - 2$. Since $\Delta \in \mathcal{D}$, $\mathcal{D}$ does not contain any number less than $\frac{p-\epsilon pq}{p+\epsilon pq}\Delta \geq \frac{1-\epsilon}{1+\epsilon}\Delta > \frac{2}{3}(k_0+1)$. Hence $n_d = 0$ for $d < \frac{2}{3}(k_0+1)$ and in particular for $d < 5$. We now require $n_d \leq b_d$ for $\frac{2}{3}(k_0+1) \leq d \leq k_0$ where $b_d = \frac{d-4}{d-3}\min\left\{2\binom{\Delta-5}{d-3}, \binom{\Delta-2}{d}\right\} - 2$.

If $7 \le k_0 \le \Delta - 3$ and $5 \le d \le k_0$ then $b_d \ge \frac{1}{2}\min\left\{2\binom{k_0-2}{d-3}, \binom{k_0+1}{d}\right\} - 2$. Since $\binom{k_0-3}{d-4}, \binom{k_0}{d-1} \ge 4$, $b_d \ge c_d = \frac{1}{2}\min\left\{2\binom{k_0-3}{d-3}, \binom{k_0}{d}\right\}$. Now $\binom{k}{d} = \frac{d+1}{k-d}\binom{k}{d+1}$, so $\frac{c_d}{c_{d+1}} \ge \frac{d-2}{k_0-d} \ge 2$ provided $d \ge \frac{2}{3}(k_0+1)$. Hence, by induction, $c_{k_0-r} \ge c_{k_0}2^r = 2^{r-1}$ whenever $r \ge 0$ and $k_0 - r \in \mathcal{D}$. Thus $b_d \ge 2^{k_0-1-d}$ for $d \in \mathcal{D}$ and $d \le k_0$. Therefore since $E_3$ fails we have $n_d \le b_d$ for these values of $d$. Now all the conditions of Corollary 3 are satisfied. Thus $\chi'_s(G) = \Delta$ provided $E_1$–$E_4$ all fail for some $k_0 \ge 7$.

Let $p_d = \binom{n-1}{d}p^d q^{n-d}$ be the probability of a vertex having degree $d$ and let $p_{d,d'}$ be the probability of two distinct vertices having degrees $d$ and $d'$ respectively. Write $\frac{d}{n-1} = p + \alpha pq$ and $\frac{d'}{n-1} = p + \alpha' pq$. A simple calculation shows that

$$\begin{aligned}
p_{d,d'} &= p\binom{n-2}{d-1}p^{d-1}q^{n-d-1}\binom{n-2}{d'-1}p^{d'-1}q^{n-d'-1} + q\binom{n-2}{d}p^d q^{n-d-2}\binom{n-2}{d'}p^{d'}q^{n-d'-2} \\
&= p_d p_{d'}\left(\tfrac{dd'}{p(n-1)^2} + \tfrac{(n-1-d)(n-1-d')}{q(n-1)^2}\right) \\
&= p_d p_{d'}\left(p(1+\alpha q)(1+\alpha' q) + q(1+\alpha p)(1+\alpha' p)\right) \\
&= p_d p_{d'}(1 + \alpha\alpha' pq) \\
&\le p_d p_{d'}(1 + \epsilon^2 pq) \qquad \text{if } d, d' \in \mathcal{D} \qquad (3)
\end{aligned}$$

If $S \subset \mathbb{Z}$, let $n_S = \sum_{d \in S} n_d$ be the number of vertices with degree $d_v \in S$ and write $\mu_d = \mathbb{E}(n_d)$, $\mu_S = \mathbb{E}(n_S)$. Then if $S \subseteq \mathcal{D}$, we have

$$\begin{aligned}
\mathrm{Var}(n_S) &= n(n-1)\sum_{d,d' \in S} p_{d,d'} + \sum_{d \in S} np_d - \left(\sum_{d \in S} np_d\right)^2 \\
&\le n^2 \sum_{d,d' \in S} p_d p_{d'}(1 + \epsilon^2 pq) - n^2 \sum_{d,d' \in S} p_d p_{d'} + \sum_{d \in S} np_d \\
&= \epsilon^2 pq\mu_S^2 + \mu_S. \qquad (4)
\end{aligned}$$

Also, if $d \in \mathcal{D}$ then

$$\frac{\mu_d}{\mu_{d+1}} = \frac{p_d}{p_{d+1}} = \frac{dq}{(n-1-d)p} \le \frac{(p+\epsilon pq)q}{(q-\epsilon pq)p} \le \frac{1+\epsilon}{1-\epsilon} \le \frac{3}{2} \qquad (5)$$

Let $S = \{d \in \mathcal{D} : d \ge k_0 + 3\}$ and $\epsilon' = \mathbb{P}(d_v \notin \mathcal{D})$. For any random variable $X$ with $\mathbb{E}(X) > t$ we have by Chebechev, $\mathbb{P}(X \le t) \le \mathrm{Var}(X)/(\mathbb{E}(X) - t)^2$. If $k_0 \ge p(n-1)$ then $p_d$ and $\mu_d$ are decreasing functions of $d$ for $d \geq k_0$. Hence we have the following estimates.

$$\begin{aligned}
\mathbb{P}(E_1) &= \mathbb{P}(n_S = 0) \leq (\epsilon^2 pq\mu_S^2 + \mu_S)/\mu_S^2 = \epsilon^2 pq + \mu_S^{-1} && (6)\\
\mathbb{P}(E_2) &\leq n(n-1)\sum_{d\geq k_0,\, d\in\mathcal{D}} p_{d,d} + n(n-1)\sum_{d\geq k_0,\, d,d+1\in\mathcal{D}} p_{d,d+1} \\
&\leq n^2 p_{k_0} 2 \sum_{d\geq k_0,\, d\in\mathcal{D}} p_d(1+\epsilon^2 pq) \\
&\leq 3\mu_{k_0}(\mu_S + 3\mu_{k_0}) && (7)\\
\mathbb{P}(E_3) &\leq \sum_{r\geq 0,\, k_0-r\in\mathcal{D}} \mathbb{P}(n_{k_0-r} \geq 2^{r-1}) \leq \sum_{r\geq 0,\, k_0-r\in\mathcal{D}} \frac{\epsilon^2 pq\mu_{k_0-r}^2 + \mu_{k_0-r}}{(2^{r-1} - \mu_{k_0-r})^2} \\
&\leq \sum_{r\geq 0} \frac{(3/2)^{2r}\mu_{k_0}^2 + (3/2)^r\mu_{k_0}}{(2^{r-1} - (3/2)^r\mu_{k_0})^2} \leq \frac{\mu_{k_0}^2 + \mu_{k_0}}{(\frac{1}{2} - \mu_{k_0})^2}\sum_{r\geq 0}(\tfrac{9}{16})^r \\
&\leq \frac{16(\mu_{k_0}^2 + \mu_{k_0})}{7(\frac{1}{2} - \mu_{k_0})^2} && (8)\\
\mathbb{P}(E_4) &\leq n\epsilon' && (9)
\end{aligned}$$

Therefore we need to show that for $n$ sufficiently large, $n\epsilon'$ is small and we can choose $k_0 \geq p(n-1) \geq 7$ so that $\mu_S$ is large and $\mu_{k_0}\mu_S$ (and hence $\mu_{k_0}$) is small.

First we shall show that if $\epsilon > 0$ is small enough, then for sufficiently large $n$ we have $n\epsilon' < \epsilon$. For this we use the estimate

$$\begin{aligned}
\mathbb{P}(\tfrac{d_v}{n-1} > p + \epsilon pq) &\leq \mathbb{E}((1+\epsilon)^{d_v})(1+\epsilon)^{-(n-1)(p+\epsilon pq)} \\
&= (q + p(1+\epsilon))^{n-1}(1+\epsilon)^{-(n-1)(p+\epsilon pq)} \\
&= \Big((1+p\epsilon)(1+\epsilon)^{-(p+\epsilon pq)}\Big)^{n-1} && (10)
\end{aligned}$$

It is an easy exercise to show that $(1+p\epsilon)(1+\epsilon)^{-(p+\epsilon pq)} = \exp(-\epsilon^2 pq(1+O(\epsilon))/2)$. Hence for small enough $\epsilon$ we have $\mathbb{P}(\frac{d_v}{n-1} > p + \epsilon pq) \leq \exp(-\epsilon^2 pqn/3)$. A similar argument shows that for small $\epsilon$, $\mathbb{P}(\frac{d_v}{n-1} < p - \epsilon pq) \leq \exp(-\epsilon^2 pqn/3)$. Thus if $\frac{pn}{\log n}, \frac{qn}{\log n} \to \infty$, then $\frac{pqn}{\log n} \to \infty$ and $\epsilon' = \mathbb{P}(d_v \notin \mathcal{D}) = O(n^{-s})$ for any $s > 0$. In particular, for sufficiently large $n$ we have $n\epsilon' < \epsilon$.

Now we need to choose $k_0$. Define $\gamma_d$ by $\gamma_d = \frac{1+\epsilon}{2\epsilon}\mu_d$ when $d = \max\{d : d \in \mathcal{D}\} + 1$ and inductively define $\gamma_d = \gamma_{d+1} + \mu_d$ for smaller values of $d$. Since $\frac{\mu_d}{\mu_{d+1}} \leq \frac{1+\epsilon}{1-\epsilon}$ for $d \in \mathcal{D}$, it is easy to check inductively that $\gamma_d \geq \frac{1+\epsilon}{2\epsilon}\mu_d$ and $\gamma_{d+1} \leq \gamma_d \leq \frac{1+\epsilon}{1-\epsilon}\gamma_{d+1}$ for all $d \in \mathcal{D}$. Also, $\mu_S \leq \gamma_{k_0} \leq \mu_S + 3\mu_{k_0} + n\epsilon'(1+\epsilon)/2\epsilon < \mu_S + 3\mu_{k_0} + 1$, so it is now sufficient to find $k_0$ with $\gamma_{k_0}$ large and $\mu_{k_0}\gamma_{k_0}$ small.

When $k_0 = \lceil p(n-1)\rceil$, $\mu_{k_0} > 1$ and so $\gamma_{k_0} \geq 1/2\epsilon$. When $k_0 = \max\{d : d \in \mathcal{D}\}+1$, $\mu_{k_0} \leq n\epsilon' < \epsilon$ and so $\gamma_{k_0} \leq \mu_{k_0}/2\epsilon < 1$. Therefore $\mu_{k_0}\gamma_{k_0}^2$ varies monotonically from a value greater than $\frac{1}{4\epsilon^2}$ when $k_0 = \lceil p(n-1)\rceil$ to a value less than $\epsilon$ when $k_0 = \max\{d : d \in \mathcal{D}\} + 1$. Take $k_0$ maximal so that $\mu_{k_0}\gamma_{k_0}^2 \geq 1$. Then $\mu_{k_0}\gamma_{k_0}^2 \leq (\frac{1+\epsilon}{1-\epsilon})^3 \leq 4$. Since $\mu_{k_0} < 2\epsilon\gamma_{k_0}$ we have $2\epsilon\gamma_{k_0}^3\mu_{k_0} \geq \mu_{k_0}$ and so $\gamma_{k_0} \geq (2\epsilon)^{-1/3}$. Also $\mu_{k_0}\gamma_{k_0} \leq 4/\gamma_{k_0} \leq 4(2\epsilon)^{1/3}$. Thus $\gamma_{k_0}$ is large and $\mu_{k_0}\gamma_{k_0}$ is small as desired. □

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