Counting Regions With Bounded Surface Area

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Abstract

Define a *cubical complex* to be a collection of integer-aligned unit cubes in d dimensions. Lebowitz and Mazel (1998) proved that there are between $(C_1 d)^{n/2d}$ and $(C_2 d)^{64n/d}$ complexes containing a fixed cube with connected boundary of $(d-1)$-volume n. In this paper we narrow these bounds to between $(C_3 d)^{n/d}$ and $(C_4 d)^{2n/d}$. We also show that there are $n^{n/(2d(d-1))+o(1)}$ connected complexes containing a fixed cube with (not necessarily connected) boundary of volume n.

1 Introduction

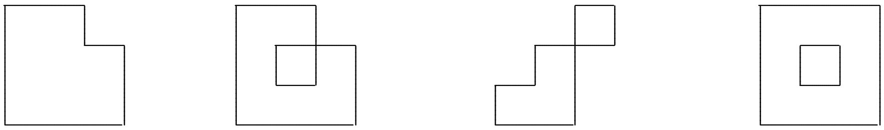
Define an *r-cube* C to be an r-dimensional unit cube in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. In other words, a set of the following form

$$C = C(\mathbf{a}, I) = \{\mathbf{x} \in \mathbb{R}^d : x_i = a_i \text{ for } i \notin I,\ a_i \le x_i \le a_i + 1 \text{ for } i \in I\},$$

where $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ and I is a subset of $\{1, \dots, d\}$ of size r. Define an *r-dimensional cubical complex* (or *r-complex*) B to be a finite union of r-cubes in $\mathbb{R}^d$. We shall call a complex *rooted* if it contains the cube $C_r = C(\mathbf{0}, \{1, \dots, r\})$. Define the *volume* $|B|$ of B to be the number of r-cubes in B.

We shall define the *boundary* ∂C of a cube C to be the $(r-1)$-complex which is the union of the r pairs of faces $C((a_1, \dots, a_i, \dots, a_d), I \setminus \{i\})$ and $C((a_1, \dots, a_i + 1, \dots, a_d), I \setminus \{i\})$ for $i \in I$. We shall also define the boundary ∂B of the complex $B = \bigcup_{i=1}^n C_i$ to be the $(r-1)$-complex which contains each $(r-1)$-cube that occurs in an odd number of boundaries ∂C_i. (We shall avoid issues of orientation in this paper.) We say B is *closed* if $\partial B = \emptyset$. Define the *surface area* of B to be the volume of the boundary $|\partial B|$.

We say that an r-complex B is *connected* if it is connected via its $(r-1)$-dimensional faces. More formally, let G be the graph with vertices equal to the component r-cubes of B and two vertices joined by an edge when these cubes share a common $(r-1)$-dimensional face. Then B is connected precisely when G is connected.



$B \in \mathcal{B}_2$	$B \in \mathcal{B}'_2$	$B \in \mathcal{B}'_2$	$B \in \mathcal{B}''_2$
B connected	B connected	B not connected	B connected
∂B primitive	∂B not primitive	∂B not primitive	∂B not connected

Figure 1: Examples of contours in 2 dimensions.

The number of d-dimensional cubical complexes with a given volume or surface area is interesting in its own right; however it also has applications to the Ising model in d dimensions, where the convergence of the low temperature expansion is dependent on the number of Peierls contours, i.e., the number of connected boundaries of rooted cubical complexes (see Lebowitz and Mazel [3]).

Following the notation of [3], we define a *contour* to be the boundary of some rooted d-complex, provided that this boundary is itself a connected $(d-1)$-complex. A contour is *primitive* if it is minimal, i.e., it is not a disjoint union of two non-empty contours. Note that, in general, if ∂B is a contour, then the cubes of B need only be connected via $(d-2)$-dimensional cubes. On the other hand, if we insist that B is itself connected, it does not follow that ∂B is a contour, since ∂B may not be connected, and even if it is, it is not necessarily primitive (see Figure 1 for some examples with $d = 2$). However, if ∂B is primitive then B must be connected (since ∂B is the disjoint union of the boundaries of the components of B).

Let $\mathcal{B}_d$ be the set of rooted d-complexes in $\mathbb{R}^d$ with primitive boundaries, $\mathcal{B}'_d$ the rooted d-complexes (possibly disconnected) with connected boundary, and $\mathcal{B}''_d$ the connected rooted d-complexes (possibly with disconnected boundary, see Figure 1). Write $S_d(n)$ (respectively $S'_d(n)$, $S''_d(n)$) for the number of elements of $\mathcal{B}_d$ (respectively $\mathcal{B}'_d$, $\mathcal{B}''_d$) with surface area n. Write $V_d(n)$ for the number of connected rooted d-complexes with volume n. Note that all these quantities are finite. In this note we shall give upper and lower bounds for all of these quantities.

2 Preliminary results

For two r-complexes, B_1 and B_2, define $B_1 \oplus B_2$ to be the complex formed from all r-cubes that are in either B_1 or B_2 but not both. Note that $\partial(B_1 \oplus B_2) = \partial B_1 \oplus \partial B_2$. Also, for each complex B and for $1 \le i \le d$, define $B_i^=$ to be the subcomplex of all r-cubes of B that have zero extent in dimension i, i.e., that are contained in some hyperplane $x_i = c$. Define $B_i^\perp$ to be the subcomplex consisting of all the r-cubes of B which have positive extent in dimension i, so that $B = B_i^\perp \oplus B_i^=$. The cubes in $B^\perp$ will be called *vertical* cubes, and the cubes in $B^{=}$ will be called *horizontal* cubes. The following slightly technical lemma will be useful.

Lemma 1. *Assume $B = B_d^{\perp}$ and $\partial B \subseteq \mathbb{R}^{d-1}$, where $\mathbb{R}^{d-1}$ is identified with the hyperplane $x_d = 0$ in $\mathbb{R}^d$. Then $B = \emptyset$.*

Proof. Assume $B \neq \emptyset$ and let $a \in \mathbb{Z}$ be the maximum integer such that B meets the hyperplane $x_d = a$. Then B contains some r-cube $C \times [a-1, a]$ with $C \subseteq \mathbb{R}^{d-1}$. The face $C \times \{a\}$ of this r-cube is a face of precisely two r-cubes in $\mathbb{R}^d$ with positive extent in dimension d. One of these is $C \times [a-1, a]$, the other is $C \times [a, a+1]$. Only the first of these is in B, so $C \times \{a\} \subseteq \partial B \subseteq \mathbb{R}^{d-1}$. Hence $a = 0$ and $B \subseteq \mathbb{R}^{d-1} \times (-\infty, 0]$. A similar argument holds for the minimal a and shows that $B \subseteq \mathbb{R}^{d-1} \times [0, \infty)$. Thus $B \subseteq \mathbb{R}^{d-1} \times \{0\}$, contradicting the assumption that every cube of B has positive extent in dimension d. □

Lemma 2. *An r-complex B is closed if and only if $B = \partial B'$ for some B'.*

Proof. Since each $(r-2)$-dimensional subcube of an r-cube C is contained in precisely two faces of C, $\partial\partial C = \emptyset$. Hence ∂B is closed for all B. We now prove the converse. For each cube $C \times \{a\}$ in $B_d^{=}$ with $a \neq 0$, construct the *stack* $C \times [0, a]$ (or $C \times [a, 0]$ if $a < 0$). The $\oplus$-sum of all these stacks is a complex E with $(\partial E)_d^{=}$ agreeing with $B_d^{=}$ outside $\mathbb{R}^{d-1}$. Let $F = B \oplus \partial E$. Then $F_d^{=} \subseteq \mathbb{R}^{d-1}$. Now F is closed so $\partial(F_d^{\perp}) = \partial(F_d^{=}) \subseteq \mathbb{R}^{d-1}$. Hence, by Lemma 1, $F_d^{\perp} = \emptyset$ and $F = F_d^{=} \subseteq \mathbb{R}^{d-1}$ is a closed complex in $\mathbb{R}^{d-1}$. By induction on d it is equal to $\partial F'$ for some F'. Now $B = \partial(E \oplus F')$ as required. □

Lemma 3. *If B and B' are two d-complexes and $(\partial B)_d^{=} = (\partial B')_d^{=}$, then $B = B'$.*

Proof. Let $E = B \oplus B'$. Then $(\partial E)_d^{=} = \emptyset$, so $\partial E = (\partial E)_d^{\perp}$. Also $\partial\partial E = \emptyset \subseteq \mathbb{R}^{d-1}$, so by Lemma 1, $\partial E = \emptyset$. But $E_d^{\perp} = E$ since every d-cube has positive extent in dimension d. Hence by Lemma 1 again, $E = \emptyset$, and so $B = B'$. □

Lemma 3 implies that the boundary ∂B of a d-complex determines the complex B. Hence counting contours is equivalent to counting elements of $\mathcal{B}'$, while counting primitive contours is equivalent to counting elements of $\mathcal{B}$.

Following [3], we construct a *floor-stack* (multi-)graph G of the boundary B of a d-complex as follows. Decompose $B_d^{=}$ as a union of connected components or *floors* F_i. Decompose $B_d^{\perp}$ into a union of complexes of the form $E_j = C \times [a, b]$ where C is an $(d-2)$-cube in $\mathbb{R}^{d-1}$ and $a, b \in \mathbb{Z}$ with $b - a$ maximal. In other words, we group together the component cubes of $B_d^{\perp}$ as maximal *stacks* of cubes in the dth dimension. Since $\partial B = \emptyset$, $C \times \{a\}$ and $C \times \{b\}$ must lie in $\partial(B_d^{=}) = \partial(B_d^{\perp})$, and hence in some ∂F_i and $\partial F_{i'}$. In other words, E_j joins F_i and $F_{i'}$. Let the vertices of G be the floors F_i and join F_i and $F_{i'}$ whenever there is a stack E_j joining F_i and $F_{i'}$.

Lemma 4. *If $B \in \mathcal{B}_d$ then the floor-stack graph of ∂B is a connected graph.*

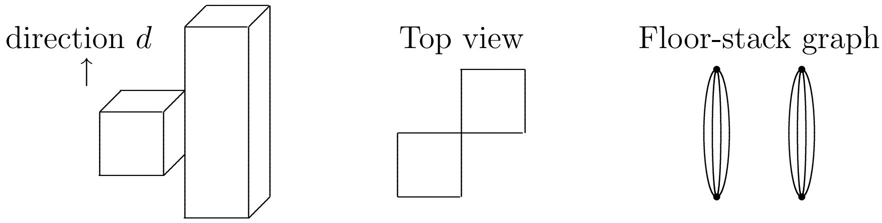


Figure 2: Example of contour with disconnected floor-stack graph.

Proof. Let E be the union of the floors F_i and stacks E_j in one component of the graph G. Let C be a horizontal $(d-2)$-cube of ∂E. Now C lies in the boundary of four $(d-1)$-cubes, two horizontal, and two vertical. If C lies in ∂E_j, then precisely one of these vertical $(d-1)$-cubes lies in ∂B. But then one of the horizontal $(d-1)$-cubes must also lie in ∂B, since otherwise C would lie in $\partial\partial B = \emptyset$. Thus C lies in the boundary of some F_i. But this F_i is then an endvertex of E_j, so also lies in the chosen component of G. But then $C \notin \partial E$, a contradiction. Similarly, if C lies in the boundary of an F_i, then it lies at the end of a stack E_j in the same component of G, once again leading to a contradiction. Hence $(\partial E)_d^{=} = \emptyset$, and thus $\partial E = (\partial E)_d^{\perp}$. Since $\partial\partial E = \emptyset \subseteq \mathbb{R}^{d-1}$, Lemma 1 implies $\partial E = \emptyset$. Thus E is a contour that is contained in ∂B. Since ∂B is primitive, $E = \partial B$ and so G is connected. □

Note that the floor-stack graph may be disconnected for non-primitive contours. See Figure 2 for an example in 3 dimensions.

3 Bounds for $V_d(n)$

We start with the easiest quantity to estimate, namely $V_d(n)$, since this illustrates some of the techniques that we shall use for the other quantities.

Theorem 5. *For all $n \geq 1$,*

$$d^{n-1} \leq V_d(n) \leq \tfrac{1}{2d-1}(2ed)^n.$$

Proof. The cube C_d is connected to $2d$ other d-cubes. For each of these choose an affine transformation that maps C_d onto this d-cube. We can construct any connected rooted d-complex by gluing smaller complexes onto C_d at some or all of the adjacent d-cubes via their root cubes using the affine transformations defined above. If we define the polynomial $f_L(X)$ inductively by $f_0(X) = X$ and

$$f_{L+1}(X) = X(1 + f_L(X))^{2d},$$

then the coefficient $a_{n,L}$ of X^n in f_L is an upper bound on the number of complexes of volume n that can be constructed by the above process in at most L steps (i.e., complexes for which every cube is within graph-distance L of the root cube). As L increases $a_{n,L}$ increases, and for $L \geq n$, $a_{n,L}$ is constant, say $a_{n,L} = a_n$. Thus $f_L(X)$ increases monotonically to $f(X) = \sum a_n X^n$ provided X is within the radius of convergence of this limiting series. Hence the number of these of volume n is bounded above by the coefficient a_n in the generating function $f(X) = \sum_{n=1}^{\infty} a_n X^n$ where $f(X)$ satisfies the equation

$$f(X) = X(1 + f(X))^{2d} \tag{1}$$

Rewrite equation (1) as $X = f(1+f)^{-2d}$ and maximize X. If the maximal $X = X_c$ occurs at $f = f_c$ then one sees inductively that $f_L(X_c) \leq f_c$ for all L. Hence the generating function $f(X)$ converges for all $X \leq X_c$. By logarithmic differentiation,

$$\frac{1}{X}\frac{dX}{df} = \frac{1}{f} - \frac{2d}{1+f}$$

so at $f = f_c$, $\frac{1}{f_c} = \frac{2d}{1+f_c}$. Thus $f_c = \frac{1}{2d-1}$ and

$$X_c = (2d-1)^{2d-1}(2d)^{-2d} = (1 + \tfrac{1}{2d-1})^{-(2d-1)}(2d)^{-1} \geq (2ed)^{-1}.$$

Therefore

$$V_d(n) \leq \sum_{i=1}^{n} a_i \leq f(X_c)X_c^{-n} \leq \frac{(2ed)^n}{2d-1}.$$

For the lower bound, note that for each sequence $(d_2, \dots, d_n)$ with $d_i \in \{1, 2, \dots, d\}$ we can construct a complex by taking a sequence of d-cubes with the ith cube located one step in the positive d_ith direction from the $(i-1)$st cube. This gives d^{n-1} distinct connected complexes. □

4 Bounds for $S_d(n)$

We start with an upper bound for the number of primitive contours with given $(d-1)$-dimensional volume. Note that this volume is always even since the surface area of each cube is even.

Theorem 6. *For all $d \geq 2$ and even $n \geq 2d$,*

$$S_d(n) \leq \tfrac{n}{8d^3}(8e^2d^2)^{n/d} \leq (8d)^{2n/d}.$$

Proof. Let $B = \partial B'$ be a primitive contour. Then, by Lemma 4, the floor-stack graph G of B is connected. Fix a spanning tree of G. Then we can reconstruct the floors by specifying each floor as a rooted $(d-1)$-complex together with connecting stacks. We can obtain an upper bound for the number of primitive contours containing the cube C_{d-1} by alternately growing floors and stacks. We shall define a generating function $g(X,Y) = \sum a_{r,s} X^r Y^s$

where $a_{r,s}$ will bound the number of possible spanning trees with total stack size s and total floor volume r. We define g by

$$g(X,Y) = X(1+\kappa g(X,Y))^{2(d-1)} \tag{2}$$

where

$$\kappa = \tfrac{4Y}{1-Y} + 1. \tag{3}$$

To see that this gives an upper bound, consider growing a complex starting with C_{d-1}. For each of the $2(d-1)$ faces of C_{d-1} we can either attach nothing, attach the neighboring horizontal $(d-1)$-cube (extending the current floor), or attach a stack, together with a horizontal $(d-1)$-cube at the other end of the stack. Note that we can never attach two stacks (since together they would form a single stack) or a stack and a horizontal cube (since then the stack would not lie in the boundary of the floor). In the cases when we attach cubes, we continue building the complex from the new horizontal $(d-1)$-cube. If we attach a stack, then the stack can go in one of two directions (up or down) and the horizonal $(d-1)$-cube at the other end of the stack can be attached in one of two positions. The stack can be any positive integral length. Hence we get a factor of $4(Y+Y^2+Y^3+\dots) = 4Y/(1-Y)$. Adding one to include the possibility of extending the floor, we get a factor of κg for each face of C_{d-1} that we add something to.

If we define $g_0(X,Y) = 0$ and $g_{L+1} = X(1+\kappa g_L)^{2(d-1)}$ then g_L is a polynomial in X with each coefficient a polynomial in Y divided by some power of $1-Y$. If $0 < Y < 1$ then the coefficients increase and stabilize at the corresponding coefficients of g. Hence, as before, g converges provided $X \le X_c$, where X_c is the maximum value of $g/(1+\kappa g)^{2(d-1)}$. This maximum occurs at $g = g_c = 1/((2d-3)\kappa)$ with

$$X_c = g_c(1+\kappa g_c)^{-2(d-1)} = \kappa^{-1}(2d-3)^{2d-3}(2d-2)^{-(2d-2)} \ge (2(d-1)\kappa e)^{-1}. \tag{4}$$

Next, we bound the number of contours containing a fixed vertical $(d-1)$-cube as the root. In this case, we grow the spanning tree of G starting with a stack. Since the root may lie in the middle of a stack, and there are floors at each end of the stack, the generating function for these is bounded by

$$\tilde{g}(X,Y) = (Y + 2Y^2 + 3Y^3 + \dots)(2g(X,Y))^2 = \frac{4Yg(X,Y)^2}{(1-Y)^2}.$$

The term $kY^k(2g(X,Y))^2$ comes from choosing the stack of length k (with k possible choices for the root). We then grow the contour starting with two floors, each of which starts in one of two directions.

Let $h(X)$ be the generating function for the number of primitive contours containing the cube C_{d-1}. Fix such a contour B. Then B contributes a term of the form X^rY^s to $g(X,Y)$ where $r = |B_d^=|$ and s is the sum of the stack lengths, in particular $s \le n = |B|$. Indeed,

B contributes many such terms, one for each spanning tree of the floor-stack graph. Now consider the above construction, except that instead of taking dimension d as vertical, take dimension i as vertical for each $i = 1, \ldots, d$. Then B contributes at least $\sum_{i=1}^{d} X^{r_i} Y^{s_i}$ to the generating function $g(X,Y) + (d-1)\tilde{g}(X,Y)$. (The root C_{d-1} is vertical in $(d-1)$ dimensions and horizontal in only one.) Since $r_i = |B_i^{=}|$, $\sum r_i = n$ and $s_i \leq n$. Thus the AM-GM inequality and the fact that $Y < 1$ gives

$$\sum_{i=1}^{d} X^{r_i} Y^{s_i} \geq d X^{n/d} Y^{\sum s_i/d} \geq d X^{n/d} Y^{n}.$$

Hence, for any $0 < Y < 1$ and $0 < X < X_c = X_c(Y)$ we have

$$g(X,Y) + (d-1)\tilde{g}(X,Y) \geq d\, h(X^{1/d} Y).$$

We wish to maximize $X^{1/d}Y$ subject to remaining inside the domain of convergence of g and $\tilde{g}$. A reasonably good choice is $Y = Y_0 = \frac{d}{d+1}$ and $X_0 = \frac{1}{8ed^2}$. Then $\kappa = 4d+1$, $X_c \geq (2e(d-1)(4d+1))^{-1} \geq X_0$, and

$$X_0 Y_0^d \geq \frac{1}{8ed^2(1+1/d)^d} \geq X_1 = \frac{1}{8e^2 d^2}.$$

Now

$$(1 + \tfrac{\kappa}{8d^2})^{2(d-1)} \leq e^{2(4d+1)(d-1)/8d^2} \leq e,$$

so

$$\tfrac{1}{8d^2}(1 + \tfrac{\kappa}{8d^2})^{-2(d-1)} \geq \tfrac{1}{8ed^2} = X_0.$$

Hence $g_0 = g(X_0, Y_0) \leq \frac{1}{8d^2}$. Also $\tilde{g}_0 = \tilde{g}(X_0, Y_0) = \frac{4d^3}{d+1} g_0^2 \leq \frac{1}{8d^2}$. Thus the number of primitive contours of size n containing C_{d-1} is at most

$$h(X_1^{1/d}) X_1^{-n/d} \leq \tfrac{1}{d}(g_0 + (d-1)\tilde{g}_0) X_1^{-n/d} \leq \tfrac{1}{8d^2}(8e^2 d^2)^{n/d}.$$

Finally, each contour surrounding C_d must contain a vertical translate of C_{d-1} at a vertical distance less than $n/(2(d-1)) \leq n/d$ below the hyperplane $x_d = 0$. Thus

$$S_d(n) \leq \tfrac{n}{8d^3}(8e^2 d^2)^{n/d}.$$

Finally for $d \geq 2$,

$$\tfrac{n}{8d^3} \leq e^{n/8d^3} \leq e^{0.04n/d},$$

so

$$\tfrac{n}{8d^3}(8e^2 d^2)^{n/d} \leq (8e^{2.04} d^2)^{n/d} \leq (8d)^{2n/d}.$$

□

Now, we turn to a lower bound on $S_d(n)$.

Theorem 7. *For all $d \geq 2$ and all even $n \geq 4d^2$ we have*

$$S_d(n) \geq d^{\frac{n-4d^2}{2(d-1)}} \geq (Cd)^{n/2d}$$

Proof. Let us use the procedure defined in Theorem 5 that for each sequence $(d_2, \ldots, d_{k+1})$ with $d_i \in \{1, 2, \ldots, d\}$ builds a complex by taking a sequence of d-cubes with the ith cube located one step in the positive d_ith direction from the $(i-1)$st cube. This gives d^k distinct connected complexes with surface area $2(k+1)(d-1)+2$. To get an arbitrary even surface area, add a $2 \times j \times 1 \times \cdots \times 1$ block in one of the negative directions. This increases the surface area by $j(4d-6)+4-2$ (the -2 is due to the loss of the joining face). Thus we obtain a complex with surface area $2(k+1+2j)(d-1)+4-2j$. If we choose j so that $4-2j \equiv n \bmod 2(d-1)$ then one can solve $n = 2(k+1+2j)(d-1)+4-2j$ for k. We can choose j so that $1 \leq j \leq d-1$, so $n \leq 2k(d-1)+4(d-1)^2+4 \leq 2k(d-1)+4d^2$. The result follows. □

Combining the upper and lower bounds we see that

$$(C_1 d)^{n/2d} \leq S_d(n) \leq (C_2 d)^{2n/d}$$

for sufficiently large even n.

5 Bounds for $S'_d(n)$

Now we extend the results of the previous section to count all contours, rather than just primitive ones.

Theorem 8. *For all $d \geq 2$ and large even n,*

$$S'_d(n) \leq \tfrac{n}{8d^3}(8e^{17/8}d^2)^{n/d} \leq (9d)^{2n/d}$$

Proof. As in Theorem 6, let $h(X)$ be the generating function for the number of primitive contours of volume n containing C_{d-1}. Fix a primitive contour containing n $(d-1)$-cubes. Then there are a total of (at most) $(d-1)n$ common $(d-2)$-cubes, since each $(d-2)$ cube occurs as the face of (at least) two $(d-1)$-cubes, and each $(d-1)$-cube has $2(d-1)$ faces. An arbitrary contour can be obtained by attaching a contour to some of the $(d-2)$-cube boundaries of the component cubes of some primitive contour. The way in which this attachment is done is essentially unique, since there are only two possible other $(d-1)$-cubes that meet this $(d-2)$-cube, and both must be in the attached contour. By a suitable ordering of all $(d-1)$-cubes in $\mathbb{R}^d$, we can fix one of these as the root of the added contour. Thus, the number of contours with volume n is bounded above by the coefficient of X^n in

$$f(X) = h(X(1+f(X))^{d-1}).$$

From the previous section we know that $h((8e^2d^2)^{-1/d}) \leq \frac{1}{8d^2}$. Since

$$(8e^2d^2)^{-1/d}/(1+\tfrac{1}{8d^2})^{d-1} = (8e^2d^2(1+\tfrac{1}{8d^2})^{d(d-1)})^{-1/d} \geq (8e^{17/8}d^2)^{-1/d},$$

if we set $X_2 = (8e^{17/8}d^2)^{-1/d}$ then $f(X_2)$ converges and $f(X_2) \leq \frac{1}{8d^2}$. Thus the number of contours of surface area (at most) n containing C_{d-1} is bounded by

$$f(X_2)X_2^{-n} \leq \tfrac{1}{8d^2}(8e^{17/8}d^2)^{n/d}.$$

As before, any contour surrounding C_d must contain one of n/d translates of C_{d-1}, so

$$S'_d(n) \leq \tfrac{n}{8d^3}(8e^{17/8}d^2)^{n/d}.$$

Finally for $d \geq 2$, $\frac{n}{8d^3} \leq e^{0.04n/d}$ so $\frac{n}{8d^3}(8e^{17/8}d^2)^{n/d} \leq (8e^{2.165}d^2)^{n/d} < (9d)^{2n/d}$. □

Our lower bound on $S'_d(n)$ is even easier to prove.

Theorem 9. *For all $d \geq 2$ and even $n \geq 2d^2$ we have*

$$S'_d(n) \geq \binom{d}{2}^{(n-d^2)/2d} \geq (Cd)^{n/d}$$

Proof. We can write $n = 2dk + 2(d-j)$ for some $k > j$, $0 \leq j \leq d-1$. Fix a sequence $(d_1, \ldots, d_j)$ with $d_i \in \{1, \ldots, d\}$ and a sequence $(p_{j+1}, \ldots, p_k)$, where each p_i is an unordered pair $(d_{i,1}, d_{i,2})$, $d_{i,s} \in \{1, \ldots, d\}$. Construct a complex by starting with the root C_d and adding cubes so that the $(i+1)$st cube is located one step in the positive d_ith direction from the ith cube when $i \leq j$ and is one step in both the positive $d_{i,1}$ and $d_{i,2}$ directions when $i > j$. The boundary of this complex is a contour surrounding C_d with surface area $2d(k+1) - 2j = n$. There are $d^j\binom{d}{2}^{k-j}$ such contours. Thus

$$S'_d(n) \geq d^j\binom{d}{2}^{k-j} \geq \binom{d}{2}^{k-j/2} \geq \binom{d}{2}^{(n-d^2)/2d}.$$

□

Combining the upper and lower bounds we find that

$$(C'_1d)^{n/d} \leq S'_d(n) \leq (C'_2d)^{2n/d}$$

for sufficiently large even n. Note that these bounds are 'closer' than for $S_d(n)$ since the lower bound is much larger.

6 Bounds for $S''_d(n)$

It turns out that this quantity is much larger than $S_d(n)$ or $S'_d(n)$, i.e., there are many more connected rooted complexes with a given surface area than there are contours.

Theorem 10. *For fixed $d \geq 2$ and all sufficiently large even n,*

$$S''_d(n) = n^{\frac{n}{2d(d-1)}(1+o(1))}.$$

Proof. For a lower bound, consider the large cube $[0, N+2]^d$ consisting of $(N+2)^d$ d-cubes. Remove k of the N^d central cubes which are of the form $C((a_1, \ldots, a_d))$ with $\sum_{i=1}^d a_i \equiv s \bmod 3$. For some choice of $s \in \{0, 1, 2\}$ there will be at least $N^d/3$ choices for these cubes, and hence at least $\binom{N^d/3}{k} \geq (N^d/3ek)^k$ possible resulting d-complexes. The restrictions on the cubes that are removed ensures that the resulting complex will always be connected, and will have surface area exactly $2d(N+2)^{d-1} + 2dk$. Assuming $N > 2d$, we can add cubes of the form $C((-1, i, 0, 0, \ldots, 0)$, $i = 2, 4, \ldots 2j$, increasing the surface area by $(2d-2)j$. Assume n is large and even. In particular, assume $n \geq 2d(N+2)^{d-1} + 2d^2$. One can choose j and k so that

$$2dk + (2d-2)j = n - 2d(N+2)^{d-1}.$$

To do this, first choose j so that $2j \equiv -n \bmod 2d$, $0 \leq j < d$. Then solve for $k = k(N)$. Finally, choose N so that $n \approx 2d(\log N)N^{d-1}$. Then $k \approx (\log N - 1)N^{d-1}$ and $(N^d/3ek)^k = n^{\frac{n}{2d(d-1)}(1+o(1))}$. This shows that $S''_d(n)$ is at least as large as claimed.

It is somewhat harder to prove a good upper bound. Note that $\mathbb{R}^d \setminus B$ has precisely one infinite component, and the boundary of this component is a contour $\partial B'$ with $B \subseteq B'$. Thus ∂B is obtained by fixing a contour $\partial B'$ and then adding some contours inside B'. Given $n_0 = |\partial B'| \leq n$, there are at most $(9d)^{2n_0/d}$ choices for B', and each such B' has volume at most $v = (n_0/2d)^{d/(d-1)}$ with equality if B' is a large cube. We shall add k contours, each of which must surround some cube of B'. There are $\binom{v+k-1}{k}$ choices for these root cubes (we allow a root cube to be chosen more than once). The added contours will be of sizes $n_1, \ldots, n_k$ where $n_1 + \cdots + n_k = n - n_0$. Since each $n_i \geq 2d$, there are $\binom{n-n_0-2dk+k-1}{k-1}$ choices for the n_i, $i > 0$ (we need to partition the 'excess' $r = n - n_0 - 2dk$ as the sum of k numbers). Now each contour can be chosen in at most $(9d)^{2n_i/d}$ ways. Thus

$$S''_d(n) \leq \sum_{k,n_0:2dk+n_0 \leq n} (9d)^{2(n_0+n_1+\cdots+n_k)/d} \binom{v+k-1}{k} \binom{n-n_0-2dk+k-1}{k-1}.$$

Now $(9d)^{2(n_0+\cdots+n_k)/d} = (9d)^{2n/d} = n^{o(n)}$, $\binom{m}{r} \leq (m/r)^r$, and there are at most $n^2 = n^{o(n)}$ choices for (k, n_0). Hence

$$S''_d(n) \leq \sum_{k,n_0:2dk+n_0 \leq n} \binom{v+k}{k} \binom{n-n_0-2dk+k}{k} n^{o(n)} \leq \max_{2dk+n_0 \leq n} \left(\frac{(v+k)(n-n_0-2dk+k)}{k^2}\right)^k n^{o(n)}.$$

We bound $v = (n_0/2d)^{d/(d-1)} \leq n^{1/(d-1)} n_0$, and $n_0 \leq n - 2dk$. Thus $v + k \leq n^{1/(d-1)}(n - 2dk) + k \leq n^{1/(d-1)}(n - 2dk + k)$ and so

$$(v+k)(n - n_0 - 2dk + k) \leq n^{1/(d-1)}(n - 2dk + k)^2.$$

This shows that

$$S_d''(n) \leq \max_{2dk \leq n} \left(\tfrac{n^{1/(d-1)}(n-2dk+k)^2}{k^2} \right)^k n^{o(n)}.$$

Now we maximize over k. Taking logarithms, we need to maximize $\frac{k \log n}{d-1} + 2k \log(n - 2dk + k) - 2k \log k$. Differentiating with respect to k gives

$$\tfrac{\log n}{d-1} + 2 \log \tfrac{n-2dk+k}{k} - \tfrac{2(2d-1)k}{n-2dk+k} - 2.$$

But $k \leq n - 2dk + k$, so for sufficiently large n this is always positive. Hence the maximum is attained for the maximum possible $k = n/2d$. Substituting this we get

$$S_d''(n) \leq n^{k/(d-1)+o(n)} = n^{n/(2d(d-1))+o(n)}.$$

□

7 Polymer expansion for the Ising model

One application of our bounds is to the convergence of the low temperature expansion of the d-dimensional Ising model in terms of Peierls contours (see [3]). The general result of Kotecký and Preiss [2] (see also Dobrushin [1] and Scott and Sokal [4]) about the convergence of cluster expansion implies the following assertion (see also Lemma 2.1 of [3]).

Lemma 11. *The polymer expansion constructed for the Ising model in terms of Peierls contours is convergent at inverse temperature β if there exists a positive function $a(\gamma)$ such that, for any contour γ,*

$$\sum_{\gamma'} e^{-\beta|\gamma'| + a(\gamma')} \leq a(\gamma),$$

where the sum is taken over all contours γ' that intersect γ.

Using our results on $S_d'(n)$ we can improve considerably the Lebowitz-Mazel bound on β implying the convergence of the polymer expansion.

Theorem 12. *The polymer expansion constructed for the Ising model in terms of Peierls contours converges at inverse temperature β for all $\beta \geq \frac{2}{d} \log(11d)$.*

Proof. Each γ' that intersects γ must have some common $(d-2)$-cube with γ. Fixing this $(d-2)$-cube C, γ' is forced to contain at least one of the four $(d-1)$-cubes meeting C. Since there are (at most) $(d-1)|\gamma|$ $(d-2)$-cubes in γ, it is enough to show

$$4(d-1)\sum_{\gamma} e^{(\alpha-\beta)|\gamma|} \le \alpha,$$

where we have chosen $a(\gamma) = \alpha|\gamma|$ and the sum is over all contours containing a fixed $(d-1)$-cube. In other words, we need to show that

$$4(d-1)\sum_{n=1}^{\infty} c_n e^{(\alpha-\beta)n} \le \alpha,$$

where c_n is the number of rooted contours with surface area n. If $e^{\alpha-\beta} \le X_2$, then from the proof of Theorem 8, $\sum_{n=1}^{\infty} c_n e^{(\alpha-\beta)n} \le \frac{1}{8d^2}$. Thus we can take $\alpha = \frac{1}{2d}$, provided β is at least

$$\alpha - \log X_2 = \tfrac{1}{2d} + \tfrac{1}{d}\log(8e^{17/8}d^2) = \tfrac{1}{d}\log(8e^{21/8}d^2) \le \tfrac{2}{d}\log(11d).$$

□

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