Vertex-Distinguishing edge colorings of graphs

P.N. Balister O.M. Riordan R.H. Schelp

September 6, 2002

Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152 USA

Abstract

We consider lower bounds on the the vertex-distinguishing edge chromatic number of graphs and prove that these are compatible with a conjecture of Burris and Schelp [8]. We also find upper bounds on this number for certain regular graphs G of low degree and hence verify the conjecture for a reasonably large class of such graphs.

1 Introduction

Let G be a simple graph with n vertices. For $d \geq 0$ write n_d for the number of vertices in G of degree d. Let $\chi'(G)$ be the minimum number of colors required in a proper edge-coloring of G. By Vizing's Theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. If we have such a proper coloring with colors $\{1, \ldots, k\}$ and v is a vertex of G, denote by $S(v)$ the set of colors used to color the edges incident to v.

A proper edge coloring of a graph is said to be *vertex-distinguishing* if each pair of vertices is incident to a different set of colors. In other words, $S(u) \neq S(v)$ whenever $u \neq v$. A vertex-distinguishing proper edge coloring will also be called a *strong* coloring. A graph has a strong coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec*-graph. The minimum number of colors required for a strong coloring of a vdec-graph G will be denoted $\chi'_s(G)$.

The concept of vertex-distinguishing colorings has been considered in several papers [1,3–5,8–11]. In [8] Burris and Schelp made the following conjecture:

Conjecture 1 *Let G be a (simple) vdec-graph and let $k = k(G)$ be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) = k$ or $k + 1$.*

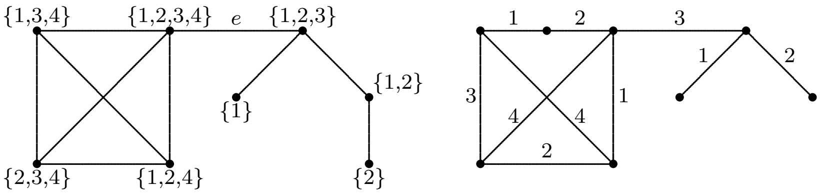
Conjecture 1 is known for a number of particular graphs, including complete graphs, complete bipartite graphs and many trees [8]. Recently Conjecture 1 has been proved for graphs of large maximum degree [4] and for G a union of cycles or a union of paths [3]. The most difficult cases therefore seem to occur when G has small maximum degree which is at least three.

If a strong coloring of a graph G exists with k colors then clearly $\binom{k}{d} \geq n_d$ for all d. Hence in Conjecture 1 we certainly have $\chi'_s(G) \geq k(G)$. In fact we can make a stronger assertion. If a strong k-coloring exists then each color meets an even number of vertices, and hence occurs in an even number of the sets $S(v)$. Since the symmetric difference $\bigoplus_{v\in V(G)} S(v)$ consists precisely of the elements that occur in an odd number of $S(v)$, this is equivalent to $\bigoplus_{v\in V(G)} S(v) = \emptyset$.

Definition 1 *Let $k'(G)$ be the minimum k such that there exist distinct sets $S_v \subseteq \{1,\ldots,k\}$ for $v \in V(G)$ with $|S_v| = \deg_G(v)$ and $\bigoplus_v S_v = \emptyset$.*

Note that $k'(G)$ and $k(G)$ both depend only on the degree sequence of G. It is clear from the above argument that $\chi'_s(G) \geq k'(G) \geq k(G)$.

There are many examples where $k'(G) > k(G)$ even for regular graphs (see Theorem 6 below). Hence in Conjecture 1 we can have $\chi'_s(G) > k(G)$. For regular graphs we know of no examples for which $\chi'_s(G) > k'(G)$. In particular, we know of no cases where two d-regular graphs with the same number of vertices have different strong chromatic numbers. The situation for non-regular graphs is different. Indeed, it is possible for two non-regular graphs with the same degree sequence to have different strong chromatic numbers. As an example, consider the following graphs.



$G_1\colon\ \chi'_s > k' = 4$ $\qquad$ $G_2\colon\ \chi'_s = k' = 4$

Looking at G_1, we see that it is possible to find distinct subsets $S_v \subseteq \{1,2,3,4\}$, one for each vertex v, with $|S_v| = \deg(v)$ and with each color occurring in an even number of subsets. Hence $k'(G_1) = 4$. Indeed, for each vertex v we can find a matching f of S_v to the neighbors of v with $c \in S_{f(c)}$. However, G_1 cannot be strongly colored with 4 colors. To see this, consider the four vertices to the left of the edge e. The choice of sets S_v shown above is essentially unique in that the set $\{1,2,3,4\}$ and all but one of the 3-sets must occur to the left of e. Hence there will always be exactly three colors occurring in an odd number of sets S_v with v on the left of e. But if $S_v = S(v)$ for some strong coloring, then the only color which can occur in an odd number of S_v for v on the left of e must be the color of e. Hence $\chi'_s(G_1) > 4$. On the other hand, the graph G_2 has the same degree sequence as G_1 but has a strong 4-coloring. This example motivates the following.

Definition 2 *Let $k''(G)$ be the smallest k such that for any set of vertices $X \subseteq V(G)$ there exist distinct sets $S_v \subseteq \{1,\ldots,k\}$, $v \in X$, such that $|S_v| = \deg_G(v)$ and $|\bigoplus_{v\in X} S_v| \leq |E(X,X^c)|$, where $E(X,X^c)$ is the set of edges between X and $X^c = V(G)\setminus X$.*

Arguing as for the example above, if $S_v = S(v)$ for some strong coloring of G then $|\bigoplus_{v\in X} S_v| \leq |E(X, X^c)|$ for all X. Hence $\chi'_s(G) \geq k''(G)$. In the next section we shall prove in general that for simple graphs $k''(G) \leq k(G) + 1$, which is at least consistent with Conjecture 1. By taking $X = V(G)$ we see that $k''(G) \geq k'(G)$, hence

$$\chi'_s(G) \geq k''(G) \geq k'(G) \geq k(G). \tag{1}$$

The definitions of χ'_s, k, k' and k'' all have natural generalizations to multigraphs. In particular, a *vdec-multigraph* is a multigraph without loops which has at most one isolated vertex and no component consisting of exactly two vertices. The inequalities (1) still hold, however Conjecture 1 fails in general for multigraphs.

While the second and third inequalities in (1) may sometimes be strict, we know of no example where $\chi'_s(G) > k''(G)$ even for multigraphs. Indeed, we have the following results, obtained by exhaustive computer search.

Result 1 *There is equality $\chi'_s(G) = k''(G)$ for all simple vdec-graphs with $|V(G)| \leq 11$, all 3-regular vdec-multigraphs with $|V(G)| \leq 22$ and all vdec-multigraphs with $|V(G)| + \Delta(G) \leq 15$.*

In particular Conjecture 1 holds for all simple graphs with $|V(G)| \leq 11$ and all 3-regular graphs with $|V(G)| \leq 22$. Result 1 suggests the following strengthening of Conjecture 1.

Conjecture 2 *For any vdec-multigraph G, $\chi'_s(G) = k''(G)$.*

For simple regular graphs we shall show (Theorem 6) that $k'(G) = k''(G)$ and give the exact value of this number. As a consequence we show that $\chi'_s(G)$ can be strictly bigger than $k(G)$ even for regular graphs. For non-regular graphs, the formulae for $k'(G)$ and $k''(G)$ are more complicated and these values may differ as shown by the example above.

For 2-regular graphs the results of [2] or [3] imply that the conditions in Theorem 6 are also sufficient for the existence of a strong k-coloring. Thus $\chi'_s(G) = k''(G) = k'(G)$ and Conjecture 2 holds. In this case we give a slightly simpler expression for $\chi'_s(G)$:

Theorem 2 *A simple 2-regular graph can be strongly colored with k colors if and only if*

1. *k is odd and either $n = \binom{k}{2}$ or $n \leq \binom{k}{2} - 3$, or*
2. *k is even and $n \leq \binom{k}{2} - \frac{k}{2}$.*

□

In Section 2 we prove the results for $k'(G)$ and $k''(G)$ mentioned above. In Sections 3 and 4 we shall restrict attention to d-regular graphs and consider the case when $n = n_d = |V(G)|$ is much larger than d. In Section 3 we give upper bounds on $\chi'_s(G)$ when G is regular and has many components and many 1-factors. In Section 4 we investigate the case of 3-regular graphs.

2 The values of $k'(G)$ and $k''(G)$

The following key lemma determines the possible symmetric differences of n distinct d-sets of $\{1, \ldots, k\}$.

Lemma 3 *Let $1 \le d \le k-1$, $0 \le t \le k$ and $n \ge 0$. There exist subsets $S_1, \ldots, S_n \subseteq \{1, \ldots, k\}$, each of size d, with $|\bigoplus_{i=1}^n S_i| = t$ if and only if the following conditions hold:*

1 $nd \equiv t \bmod 2$,

2 $n \ge \max(\frac{t}{d}, \frac{t^c}{k-d})$, where $t^c = t$ if n is even and $t^c = k - t$ if n is odd.

Furthermore, the S_i can be chosen as distinct *sets if and only if in addition*

3 if $n = 2$ then $t > 0$,

4 if $d = 1$ then $n = t$ and if $d = k-1$ then $n = t^c$, where $t^c = t$ if n is even and $t^c = k - t$ if n is odd.

5 Conditions 2 and 3 also hold when n is replaced by $\binom{k}{d} - n$ and t is replaced by t', where $t' = t$ if $\binom{k-1}{d-1}$ is even and $t' = k - t$ if $\binom{k-1}{d-1}$ is odd.

Proof. We first show that these conditions are necessary.
1. This follows since $t = |\bigoplus_{i=1}^n S_i| \equiv \sum_{i=1}^n |S_i| = nd \bmod 2$.
2. Write $S_i^c = \{1, \ldots, k\} \setminus S_i$ for the complement of S_i. Condition 2 holds since $t = |\bigoplus_{i=1}^n S_i| \le \sum_{i=1}^n |S_i| = nd$ and $t^c = |\bigoplus_{i=1}^n S_i^c| \le \sum_{i=1}^n |S_i^c| = n(k-d)$.
3. If $S_1 \ne S_2$ then $t = |S_1 \oplus S_2| > 0$.
4. If $d = 1$ (resp. $d = k-1$) then equality holds in the first (resp. second) part of the proof of Condition 2 since the sets S_i (resp. S_i^c) are disjoint.
5. Replace $\mathcal{S} = \{S_1, \ldots, S_n\}$ with the d-sets $\mathcal{S}' = \{S_1', \ldots, S_{n'}'\}$ that are not in $\mathcal{S}$. Then $n' = \binom{k}{d} - n$ and $\bigoplus S_i' = (\bigoplus S_i) \oplus X$ where X is the symmetric difference of all d-sets. However each $c \in \{1, \ldots, k\}$ occurs in $\binom{k-1}{d-1}$ d-sets, so $X = \emptyset$ if this is even and $X = \{1, \ldots, k\}$ if this is odd. Hence $|\bigoplus S_i'| = t'$ and the result follows.

We now show that the conditions given above are sufficient. Note that there are two symmetries involved. The first changes (n, d, t) to $(n, k-d, t^c)$ and corresponds to replacing the sets S_i with their complements S_i^c. The second changes (n, d, t) to $(\binom{k}{d} - n, d, t')$ and corresponds to replacing $\mathcal{S}$ with the collection $\mathcal{S}'$ of d-sets not in $\mathcal{S}$. Using the fact that $\binom{k}{d} = \binom{k-1}{d-1} + \binom{k-1}{d}$ it can be shown that $(t^c)' = (t')^c$ and so these symmetries commute. It can also be checked that all the conditions are symmetric under the first symmetry, (for Condition 5 we need the fact that the symmetries commute). Hence without loss of generality we may assume $d \le k/2$. The case when $d = 1$ is trivial, so we may also assume $d \ge 2$. To avoid special cases in the proof we also exclude the case $k = 4$, $d = 2$ which may easily be checked by hand. Thus we may assume $k \ge 5$. We shall prove the case when the S_i are required to be distinct. The case when they are not required to be distinct is much easier, so we shall just indicate the differences in the proof when they occur.

Pick any collection of (distinct) sets $\mathcal{S} = \{S_1, \ldots, S_n\}$ with $|\bigoplus S_i|$ as close to t as possible. Let $X = \bigoplus S_i$. If $|X| = t$ then we are done. Otherwise $|X|$ must differ from t by at least two by Condition 1. Hence we can find $a, b \in \{1, \ldots, k\}$ such that $|X \oplus \{a, b\}|$ is closer to t than $|X|$. Let α be the number of sets $S_i \in \mathcal{S}$ with $a \in S_i$, $b \notin S_i$. If $P \subseteq \{1, \ldots, k\}$ and $a, b \notin P$ then $P \cup \{a\} \in \mathcal{S}$ if and only if $P \cup \{b\} \in \mathcal{S}$ since if precisely one of these sets was in $\mathcal{S}$ we could replace it with the other. This would change X to $X \oplus \{a, b\}$ contradicting the choice of $\mathcal{S}$. Hence α is also the number of sets with $a \notin S_i$, $b \in S_i$ and the symmetric difference of the sets containing exactly one of a and b is just $\{a, b\}$ or $\emptyset$ depending on whether α is odd or even. In particular, we may change these sets to any collection $\{P_i \cup \{a\}, P_i \cup \{b\} : i = 1, \ldots, \alpha\}$ without changing X or violating the condition that the S_i be distinct. If the sets S_i are not required to be distinct, we can always replace $P_1 \cup \{a\}$, say, with $P_1 \cup \{b\}$, so in this case $\alpha = 0$. If the sets S_i are required to be distinct then it can be checked that Conditions 1–5 are symmetric under the second symmetry, and that $|X \oplus \{a, b\}|$ is still better than $|X|$. This symmetry replaces α by $\binom{k-2}{d-1} - \alpha$. Hence, as $\binom{k-2}{d-1} \geq 3$, we may assume that $\alpha \leq \binom{k-2}{d-1} - 2$.

Suppose there exist $c \in \{1, \ldots, k\}$ with $c \neq a, b$ and sets $P, Q \subseteq \{1, \ldots, k\}$ with $a, b, c \notin P, Q$ such that $P \cup \{c\}, Q \cup \{c\} \in \mathcal{S}$ and $P \cup \{a\}, Q \cup \{b\} \notin \mathcal{S}$. Then replacing the sets $P \cup \{c\}$ and $Q \cup \{c\}$ by $P \cup \{a\}$ and $Q \cup \{b\}$ will change X to $X \oplus \{a, b\}$, contradicting the choice of $\mathcal{S}$. Similarly if $P \cup \{c\}, Q \cup \{c\} \notin \mathcal{S}$ and $P \cup \{a\}, Q \cup \{b\} \in \mathcal{S}$ then we can replace $P \cup \{a\}$ and $Q \cup \{b\}$ by $P \cup \{c\}$ and $Q \cup \{c\}$ to obtain a contradiction. Assume $S, S' \in \mathcal{S}$ are two sets with $a, b \notin S, S'$ and $S \cap S' \neq \emptyset$. Let $c \in S, S'$ and set $P = S \setminus \{c\}$, $Q = S' \setminus \{c\}$. If $\alpha \leq \binom{k-2}{d-1} - 2$, then by changing the sets P_i as above we may assume P and Q are not equal to any of the P_i. We then get a contradiction by the argument above, since $P \cup \{c\}, Q \cup \{c\} \in \mathcal{S}$ and $P \cup \{a\}, Q \cup \{b\} \notin \mathcal{S}$. Hence the sets $S \in \mathcal{S}$ with $a, b \notin S$ are disjoint and there are therefore at most $\frac{k-2}{d}$ of them. A similar argument shows that the sets $S \in \mathcal{S}$ with $a, b \in S$ must have disjoint complements S^c, so there are at most $\frac{k-2}{k-d}$ such sets. In fact, since we have assumed $d \leq k/2$, there must be at most one of these sets. If $\alpha \geq 2$ we can apply the same arguments to the complement of $\mathcal{S}$ to show that there can be at most $\frac{k-2}{d}$ sets S' with $a, b \notin S'$ and $S' \notin \mathcal{S}$ and at most 1 set S' with $a, b \in S'$ and $S' \notin \mathcal{S}$. Hence there are at most $\frac{2(k-2)}{d}$ d-sets in total not containing a and b, and at most 2 d-sets containing a and b. Hence $\binom{k-2}{d} \leq \frac{2(k-2)}{d}$ and $\binom{k-2}{d-2} \leq 2$, but as $2 \leq \alpha \leq \binom{k-2}{d-1} - 2$, $\binom{k-2}{d-1} \geq 4$. These conditions are inconsistent, so we may assume $\alpha \leq 1$.

Now assume $\alpha = 1$ (and S_i are distinct). Assume first that there are d-sets $S \in \mathcal{S}$, $S' \notin \mathcal{S}$ with either $a, b \in S, S'$ or $a, b \notin S, S'$. We can change S into S' by successively swapping elements in $S \setminus S'$ with those in $S' \setminus S$. Moreover, no such change involves the elements a or b. Hence there must be elements c, c' distinct from a and b and sets $S'' \cup \{c\} \in \mathcal{S}$, $S'' \cup \{c'\} \notin \mathcal{S}$ with either $a, b \in S''$ or $a, b \notin S''$. Choosing P_1 appropriately we can assume $c \in P_1$, $c' \notin P_1$ (since $2 \leq d \leq k-2$). Replacing $S'' \cup \{c\}$ with $S'' \cup \{c'\}$ and $P_1 \cup \{a\}$ with $P_1 \cup \{b, c'\} \setminus \{c\}$ changes X to $X \oplus \{a, b\}$ contradicting the choice of $\mathcal{S}$. (The condition that the S_i are distinct is still satisfied since $\alpha = 1$.) Hence $\mathcal{S}$ must contain either all or none of the d-sets S with $a, b \in S$ and either all or none of the d-sets S with $a, b \notin S$. However, if $\mathcal{S}$ contains all sets S with $a, b \notin S$ then $\binom{k-2}{d} \leq \frac{(k-2)}{d}$ contradicting the assumptions on d and k. Similarly, if $\mathcal{S}$ contains all sets S with

$a, b \in S$ then $\binom{k-2}{d-2} \leq 1$ so $d = 2$, $\mathcal{S} = \{\{a, c\}, \{b, c\}, \{a, b\}\}$ and $t \geq 2$. Since $k \geq 5$ this $\mathcal{S}$ can be changed to $\mathcal{S} = \{\{a, c\}, \{b, c'\}, \{c, c'\}\}$ increasing $|X|$ and contradicting the choice of $\mathcal{S}$. Finally if $\mathcal{S}$ contains no set S with either $a, b \in S$ or $a, b \notin S$ then $\mathcal{S} = \{P_1 \cup \{a\}, P_1 \cup \{b\}\}$, $n = 2$, and $t = 0$ which is excluded by Condition 3.

Now assume $\alpha = 0$. Since $\alpha \leq \binom{k-2}{d-1} - 2$, the collection $\mathcal{S}$ now consists of disjoint sets S_i which do not contain a or b together with possibly one more set S_n that contains both. If all the sets in $\mathcal{S}$ do not contain a or b then $|X| = nd$. But $t > |X|$ since $|X \oplus \{a, b\}|$ is closer to t than $|X|$. This contradicts Condition 2. If $\mathcal{S}$ contains a unique set S_n with $a, b \in S_n$ then $t < |X|$. In this case, a and b could have been chosen as any two distinct elements of X. Hence any two elements of X must lie in one and the same set $S_i \in \mathcal{S}$. By fixing a and varying b, say, we see that $X \subseteq S_n$. Since the S_i, $i < n$, do not contain a and b we have from above that they are disjoint. But $\bigoplus S_i = X$, so $S_i \subseteq S_n$ for all $i < n$. Since the sets S_i are all d-sets and $S_i \neq S_n$ for $i < n$, this implies $n = 1$ and the result is then clear. $\square$

The next lemma is a minor extension of Vizing's Theorem which gives us a coloring which is "vertex-distinguishing in degree 1". Recall that $n_d(G)$ is the number of vertices of G of degree d.

Lemma 4 *If G is a simple graph which contains no isolated edges and $k = \max(\Delta(G), n_1(G)) + 1$ then there exists a proper edge k-coloring of G in which each degree 1 vertex sees a distinct color.*

Proof. Let G' be the multigraph obtained by identifying the degree 1 vertices of G. All multiple edges of G' must meet the vertex v corresponding to the degree 1 vertices of G and since G contains no isolated edges, G' will have no loops. Also $\Delta(G') = \max(\Delta(G), n_1(G))$, so a proper $(\Delta(G') + 1)$-coloring of G' would give us the required coloring of G. Thus it is enough to prove Vizing's Theorem for multigraphs G' in which there are no loops and all multiple edges meet a specified vertex v. Indeed, by considering the algorithm used in the proof of Vizing's Theorem [6], it is enough to give a proper coloring on the subgraph induced by $N(v) \cup \{v\}$ since any such coloring can be extended inductively to one on G' using exactly the same algorithm. Hence we are reduced to the problem of finding a proper $(\Delta(G')+1)$-coloring of G' when all multiple edges meet v and all other vertices are adjacent to v. Let $e_1, \ldots, e_r$ be a minimal set of edges which on removal make G' into a simple graph G''. Now $|N(v)| = \deg_{G''}(v) = \deg_{G'}(v) - r$, so $\Delta(G'') \leq \Delta(G') - r = k - 1 - r$. Hence G'' has a proper $(k - r)$-coloring. Coloring $e_1, \ldots, e_r$ with r distinct additional colors gives a proper k-coloring of G' as required. $\square$

Theorem 5 *For any simple vdec-graph, $k''(G) \leq k(G) + 1$.*

Proof. Fix $X \subseteq V(G)$. Since $k(G) \geq \max(\Delta(G), n_1(G))$, Lemma 4 gives a proper edge coloring of G using at most $k = k(G) + 1$ colors in which the vertices of degree 1 see distinct colors. Hence we have (not necessarily distinct) sets $S_v = S(v)$ with the property that $|\bigoplus_{v \in X} S_v| \leq |E(X, X^c)|$. We now require that the S_v be made distinct. Fix d and let X_d be the set of vertices of degree d that lie in X. We shall modify the S_v for $v \in X_d$ so as to make the sets distinct while preserving their symmetric difference. Doing this in each degree separately will give the result. Since $n_{k-1} =$

$n_{k(G)} \leq 1$, the sets S_v are already distinct in degrees 1 and $k-1$ so we may assume $2 \leq d \leq k-2$. Assume that $|X_d| \neq 2$. Then by Lemma 3 it is sufficient that $|X_d| \leq \binom{k}{d} - \max(\frac{k}{d}, \frac{k}{k-d}, 3)$ since $t, t^c \leq k$. But $|X_d| \leq n_d \leq \binom{k-1}{d}$, so it is sufficient that $\binom{k}{d} - \binom{k-1}{d} = \binom{k-1}{d-1} \geq \max(\frac{k}{d}, \frac{k}{k-d}, 3)$. This holds for all d and k with $2 \leq d \leq k-2$. It therefore remains to deal with the case when X contains precisely two vertices of degree d and these two vertices have the same value of S_v. If X contains any other vertex of degree $d' \neq d$ then $0 < |X_{d'}| < \binom{k}{d'}$ (since $k = k(G)+1$). If we let $\mathcal{S} = \{S_v : v \in X\}$, then there must be colors a, b and a set P with $P \cup \{a\} \in \mathcal{S}$, $P \cup \{b\} \notin \mathcal{S}$ and $|P \cup \{a\}| = d' \neq d$. By changing $P \cup \{a\}$ to $P \cup \{b\}$ and by choosing the two d-sets to be of the form $P' \cup \{a\}$ and $P' \cup \{b\}$ for some P', we can make the d-sets distinct and still have $|\bigoplus_{v \in X} S_v| \leq |E(X, X^c)|$. Hence we may assume X consists of just two vertices v, v', both of degree d. However, it is clear in this case that $|E(X, X^c)| \geq 2(d-1)$, so we are done provided we ensure that S_v and $S_{v'}$ have nonempty intersection. $\square$

We now calculate the exact values of $k'(G)$ and $k''(G)$ for regular graphs.

Theorem 6 *If G is a simple d-regular graph on n vertices with $d \geq 2$ then $k''(G) = k'(G)$ and both are equal to the smallest k such that*

(a) if $2d \leq k$ and $\binom{k-1}{d-1}$ is even then either $n = \binom{k}{d}$ or $n \leq \binom{k}{d} - 3$,

(b) if $2d \leq k$ and $\binom{k-1}{d-1}$ is odd then $n \leq \binom{k}{d} - \frac{k}{d}$,

(c) if $2d \geq k$ and $n \equiv \binom{k-1}{d} \bmod 2$ then either $n = \binom{k}{d}$ or $n \leq \binom{k}{d} - 3$,

(d) if $2d \geq k$ and $n \not\equiv \binom{k-1}{d} \bmod 2$ then $n \leq \binom{k}{d} - \frac{k}{k-d}$,

Proof. We first show that Conditions (a)–(d) hold for $k = k'(G)$ by setting $t = 0$ in Lemma 3.
(a) If $\binom{k-1}{d-1}$ is even then $t' = t = 0$ and Conditions 5 and 3 imply that $n \neq \binom{k}{d} - 2$. Also Conditions 5 and 2 imply that $n \neq \binom{k}{d} - 1$ since Condition 2 holds with $n = 1$ and $t^c = k$.
(b) If $\binom{k-1}{d-1}$ is odd, then $t' = k$ and Conditions 5 and 2 imply that $n \leq \binom{k}{d} - \frac{k}{d}$.
(c) Since $\binom{k}{d} = \binom{k-1}{d} + \binom{k-1}{d-1}$, we have $\binom{k}{d} - n \equiv \binom{k-1}{d-1} \bmod 2$. Part (a) implies $n \neq \binom{k}{d} - 2$ (since $\binom{k-1}{d-1}$ would be even) and part (b) implies $n \neq \binom{k}{d} - 1$ (since $\binom{k-1}{d-1}$ would be odd and $k > d$).
(d) If $\binom{k}{d} - n \not\equiv \binom{k-1}{d-1} \bmod 2$ then Conditions 5 and 2 imply $(t')^c = k$ and $n \leq \binom{k}{d} - \frac{k}{k-d}$.
Note that the assumptions $2d \leq k$ and $2d \geq k$ are not required in this part of the proof.

Now let k be the smallest k satisfying the conditions above. It is enough to show $k''(G) \leq k$ since we already know that $k''(G) \geq k'(G) \geq k$. It can be checked that $\chi'_s(K_r) = k(K_r) = r$ when r is odd [8], so the theorem is true in this case. For all other graphs G, $k \geq d+2$, so we can assume $2 \leq d \leq k-2$. (If $k \leq d$ then $n \leq \binom{k}{d} \leq 1$ and if $k = d+1$ then $n \leq d+1$ so G is a complete graph K_{d+1}. But then $\binom{k-1}{d} = 1$, so by (d) $n = d+1$ must be odd.) Fix d and k and let t_r be the smallest value of t that satisfies the conditions of Lemma 3 when $n = r$. To show $k \geq k''(G)$ it is enough that $t_r \leq |E(X, X^c)|$ for all r and X with $X \subseteq V(G)$ and $|X| = r$. We therefore need to show that $t_r \leq |E(X, X^c)|$ for all such r and X under Conditions (a)–(d) of the theorem.

Translating the conditions of Lemma 3 and observing that t_r is well defined for $0 \le r \le \binom{k}{d}$ we obtain that t_r is the smallest non-negative integer t such that

1. $t \equiv rd \bmod 2$,
2. $t \ge k - r(k-d)$ if r is odd,
3. $t > 0$ if $r = 2$,

$2'$. $t \ge k - r'd$ if $\binom{k-1}{d-1}$ is odd,

$2''$. $t \ge k - r'(k-d)$ if $r \not\equiv \binom{k-1}{d} \bmod 2$,

$3'$. $t > 0$ if $r' = 2$ and $\binom{k-1}{d-1}$ is even,

where $r' = \binom{k}{d} - r$. All the other conditions on t give upper bounds which we are not interested in. Condition 5 of Lemma 3 followed by Conditions 2 and 3 gives rise to $2'$, $2''$ and $3'$. In Condition $2''$, $r \not\equiv \binom{k-1}{d} \bmod 2$ is equivalent to either $\binom{k-1}{d-1}$ odd or r' odd but not both, so $(t')^c = k - t$. We shall show that $t = |E(X, X^c)|$ satisfies all of these conditions. This will imply $t_r \le |E(X, X^c)|$ as required. Recall that k is the smallest integer satisfying (a)–(d) and that $k \ge d+2$ and $|X| = r$.
Condition 1 — $|E(X, X^c)| \equiv rd \bmod 2$.
This follows since the sum of the degrees of the vertices in X is $2|E(X)| + |E(X, X^c)| = |X|d = rd$.
Condition 2 — $|E(X, X^c)| \ge k - r(k-d)$ if r is odd.
Indeed, if $r > k$ then $k - r(k-d) < 0$ and if $1 \le r \le k$ then $|E(X, X^c)| \ge r(d-r+1) \ge k - r(k-d)$.
Condition 3 — $|E(X, X^c)| > 0$ if $r = 2$.
Once again, $|E(X, X^c)| \ge r(d-r+1) = 2(d-1) > 0$ since $d \ge 2$.
Condition $2'$ — $|E(X, X^c)| \ge k - r'd$ if $\binom{k-1}{d-1}$ is odd.
If $2d \le k$ then we are in case (b), so $r' \ge \frac{k}{d}$ and $2'$ holds. Otherwise $2d \ge k$ and so $k - r'd$ is only positive when $r' = 0$ or 1. In these cases $n \ge \binom{k}{d} - 1$, so we must be in case (c) and $n = \binom{k}{d} \equiv \binom{k-1}{d} \bmod 2$ contradicting the fact that $\binom{k-1}{d-1}$ is odd.
Condition $2''$ — $|E(X, X^c)| \ge k - r'(k-d)$ if $r \not\equiv \binom{k-1}{d} \bmod 2$.
If $2d \ge k$ then we are in case (d), so $r' \ge \frac{k}{k-d}$ and $2''$ holds. Otherwise $2d \le k$ and so $k - r'(k-d)$ is only positive when $r' = 0$ or 1. In these cases $n \ge \binom{k}{d} - 1$, so we must be in case (a), $\binom{k-1}{d-1}$ is even, $n = \binom{k}{d}$, $r = n-1$ (by parity) and $r' = 1$. But then $|E(X, X^c)| = d = k - r'(k-d)$.
Condition $3'$ — $|E(X, X^c)| > 0$ if $r' = 2$ and $\binom{k-1}{d-1}$ is even.
If $r' = 2$ then either $n = \binom{k}{d}$ or $k = 2d$, $n = \binom{k}{d} - 2$. In the first case $|E(X, X^c)| \ge 2(d-1) > 0$. The second case contradicts (a). $\square$

3 Regular graphs with small components

Define an integer valued function $f(k, d, M)$ inductively for all $k > d \ge 2$ and $M \ge 0$ by

$$f(k,d,M)=\begin{cases}\binom{k}{2}-3 & \text{if } d=2,\ k \text{ is odd,}\\ \binom{k}{2}-\frac{k}{2} & \text{if } d=2,\ k \text{ is even,}\\ k-1 & \text{if } k=d+1>3,\\ f(k-1,d,M)+f(k-1,d-1,M)-(M-1) & \text{if } k>d+1>3.\end{cases}$$

Theorem 7 *Let $d \geq 2$ and assume G is a simple d-regular graph which contains $d-2$ disjoint 1-factors. Let $G=\bigcup_{i=1}^{s} G_i$ be the decomposition of G into components. If $\max_i |V(G_i)| \leq M$ and $|V(G)| \leq f(k,d,M)$ then G can be strongly colored with k colors.*

Proof. The proof is by induction on d. The case $d=2$ follows immediately from Theorem 2. Now assume $d>2$ and prove the result by induction on k. If $k=d+1$ then $|V(G)| \leq k-1=d$. However, this is impossible since G is d-regular and so contains at least $d+1$ vertices. Hence the result holds vacuously for $k=d+1$. Now assume $k>d+1>3$ and the result holds for smaller values of k or d.

Pick a maximal subset of components, $\bigcup_{i=1}^{r} G_i$ say, with total number of vertices at most $f(k-1,d-1,M)$. Pick one of the 1-factors I of G. Removing the edges of I from $\bigcup_{i=1}^{r} G_i$ will give a $(d-1)$-regular graph G' satisfying the conditions of the lemma with $k-1$ colors. Hence we can strongly color it with $k-1$ colors. Coloring the edges of I that lie in $\bigcup_{i=1}^{r} G_i$ with the k'th color gives a strong coloring of $\bigcup_{i=1}^{r} G_i$ in which color k meets every vertex. Now inductively color the rest of the graph with the first $k-1$ colors. This succeeds since either $\bigcup_{i=1}^{r} G_i = G$ or $\sum_{i=1}^{r} |V(G_i)| \geq f(k-1,d-1,M)-(M-1)$ and so the remaining graph has at most $f(k-1,d,M)$ vertices. Also, no vertex outside $\bigcup_{i=1}^{r} G_i$ meets color k so the coloring on the whole of G is strong. □

Corollary 8 *Let $d \geq 3$ and assume G is a simple d-regular graph which contains $d-2$ disjoint 1-factors. As before write $G=\bigcup_{i=1}^{s} G_i$ as a union of components G_i. If $|V(G)| \leq \binom{k}{d}$ and $|V(G_i)| \leq \frac{3(k-1)}{4(d-1)}$ for all i then G can be strongly colored with $k+1$ colors.*

Proof. Since $|V(G_i)| \geq d+1$, we may assume $k \geq 4(d^2-1)/3+1 \geq d+1$. By Theorem 7 it is sufficient to prove $f(k+1,d,M) \geq \binom{k}{d}$ for $M \leq \frac{3(k-1)}{4(d-1)}$. Indeed, since $f(k,d,M)$ (extended to non-integral M) is a decreasing function of M we can assume $M=\frac{3(k-1)}{4(d-1)}$. First we shall give a general bound for $f(k,d,M)$. Define $\delta_{k,d}$ by the formula

$$f(k,d,M)=\binom{k}{d}-\frac{1}{4}\binom{k-1}{d-1}-\left(M+\frac{3}{4}\right)\binom{k-3}{d-2}+\delta_{k,d}+(M-1).$$

The definition of $f(k,d,M)$ above then implies

$$\delta_{k,d}=\begin{cases}(k-6)/4 & \text{if } d=2,\ k \text{ odd,}\\ (6-k)/4 & \text{if } d=2,\ k \text{ even,}\\ (k+2)/4 & \text{if } k=d+1>3,\\ \delta_{k-1,d}+\delta_{k-1,d-1} & \text{if } k>d+1>3.\end{cases}$$

In particular, $\delta_{k,d}$ is independent of M. For $d = 3$, it can be proved by induction that $\delta_{k,3} = \frac{21-k}{8}$ if k is odd and $\frac{k+8}{8}$ if k is even. Similarly $\delta_{k,4} = \frac{15k-61}{8}$ if k is odd and $\frac{7k-27}{4}$ if k is even. In particular, $\delta_{k,4} \geq 1$ for all $k \geq 5$. Hence by induction $\delta_{k,d} \geq 1$ for all $k > d \geq 4$. Thus if $M = \frac{3(k-1)}{4(d-1)}$ then $\delta_{k+1,d} + (M-1) \geq 0$ for all $k \geq d \geq 3$ where for $d = 3$ we have used the explicit formulae for $\delta_{k,3}$. It is therefore enough to show

$$\binom{k+1}{d} - \frac{1}{4}\binom{k}{d-1} - \left(M + \frac{3}{4}\right)\binom{k-2}{d-2} \geq \binom{k}{d}.$$

Since $\binom{k+1}{d} = \binom{k}{d} + \binom{k}{d-1}$, this is equivalent to

$$\left(M + \frac{3}{4}\right)\binom{k-2}{d-2} \leq \frac{3}{4}\binom{k}{d-1}$$

or equivalently

$$\frac{3(k+d-2)}{4(d-1)} \leq \frac{3k(k-1)}{4(d-1)(k-d+1)}.$$

Since $(k+d-2)(k-d+1) = k(k-1) - (d-1)(d-2) \leq k(k-1)$, this is true for all k. The result follows. $\square$

In particular, Corollary 8 proves Conjecture 1 for d-regular graphs with $d-2$ 1-factors and sufficiently small components.

4 3-regular graphs

For this section we shall assume that G is a 3-regular multigraph. Note that for a 3-regular multigraph to be a vdec-multigraph it is necessary and sufficient that G contains no triple edges and no loops. Although Conjecture 1 is false in general for arbitrary multigraphs, it can be extended to 3-regular multigraphs, and we still know of no counterexamples.

If S is a set of independent edges of G, define $H_S = (G \setminus S)[2]$ to be the subgraph of G induced by the degree 2 vertices of $G \setminus S$, i.e., the endvertices of the edges in S. The graph H_S is then a union of paths and cycles (and double edges). Define G_S to be the graph obtained by contracting the degree 2 vertices of $G \setminus S$. If $G \setminus S$ contains a cycle, we remove this cycle completely. In other words, we contract the paths of H_S in $G \setminus S$ to a single edge and remove the cycles. The graph G_S is then a 3-regular multigraph on $n - 2|S|$ vertices which may contain loops or multiple edges even if G did not.

Theorem 9 *Assume we can find a set S of independent edges of G such that $H_S = (G \setminus S)[2]$ contains no component with fewer than 3 vertices. Assume also that H_S either consists of exactly two paths (and no cycles), or contains at most one path (and any number of cycles). Assume further that G_S can be strongly colored with k colors and $2|S| \leq f(k, 2, 0)$. Then G can be strongly colored with $k+1$ colors.*

Proof. Assume we are given a strong k-coloring of G_S. The graph G can be obtained by subdividing at most two of the edges in G_S and inserting the vertices and edges of S and H_S. Write H_S as a union of paths $P_1, \ldots, P_s$ and cycles $C_{s+1}, \ldots, C_r$ where either $s \leq 1$ or $r = s = 2$. Let $x_i y_i$ be the edge in G_S that is to be replaced by path $x_i P_i y_i$ for $i = 1, \ldots, s$. Color the edges of S with color $k+1$. It just remains to color the paths $x_i P_i y_i$ and cycles C_i. The edges meeting x_i and y_i will receive the same color that $x_i y_i$ received in G_S. The edges in P_i and C_i must now be colored with colors $1, \ldots, k$ so that each vertex of $H_S = P_1 \cup \ldots \cup P_s \cup C_{s+1} \cup \ldots \cup C_r$ sees a distinct pair of colors. This is equivalent to finding a strong k-coloring of $H' = C_1 \cup \ldots \cup C_r$ where $C_1, \ldots, C_s$ are the cycles obtained by joining the two endvertices of P_i with an edge e_i and assuming the colors of e_i are forced to be the same as those of the edges $x_i y_i$ of G_S. The condition on the size of the components of H_S ensures that all the C_i are cycles of length at least 3. By Theorem 2 we can strongly k-color H' provided $|V(H')| = 2|S| \leq f(k, 2, 0)$. It remains to show that we can choose the coloring so that $e_1, \ldots, e_s$ are colored correctly.

If $s = 1$ we can clearly permute the colors $\{1, \ldots, k\}$ to ensure e_1 receives the correct color. Hence we may assume $r = s = 2$. Assume the colors of $x_1 y_1$ and $x_2 y_2$ are distinct. By cyclically permuting the vertices of C_2, we can assume e_1 and e_2 have distinct colors. Now by permuting the colors $\{1, \ldots, k\}$ on H' we can assume they have the correct colors.

Now assume the colors on $x_1 y_1$ and $x_2 y_2$ are the same. If any edge on C_1 is colored the same as some edge on C_2, then by permuting vertices and colors we can arrange it so that e_1 and e_2 receive the correct color. If C_1 and C_2 have no color in common, change the color of e_2 to that of e_1. This new coloring is still strong and by permuting the colors we can assume e_1 and e_2 receive the correct color. Hence in all cases we are done. □

Unfortunately, a typical 3-regular graph will not have a suitable S. Even so, there are many graphs that the above algorithm will color in $k+1$ colors when $|V(G)| \leq \binom{k}{d}$. For example, the graph G_n obtained by adding the n longest diagonals to C_{2n} can be colored.

Corollary 10 *If G_n is the 3-regular graph on $2n$ vertices obtained by adding all longest diagonals to C_{2n} and if $2n \leq \binom{k}{3}$ then $\chi'_s(G_n) \leq k+1$.*

Proof. We shall prove this for $2n \leq f(k+1, 3, 2)$. The result will follow from the proof of Corollary 8 since $f(k+1, 3, 2) \geq \binom{k}{3}$ for $k \geq 5$. The proof is by induction on n. The cases G_n for $n \leq 3$ can be checked easily. For larger n let S be r consecutive diagonals where $4 \leq 2r \leq f(k, 2, 0) = f(k, 2, 2)$. As $8 \leq 2n \leq f(k+1, 3, 2)$ we can choose such an r with $4 \leq 2(n-r) \leq f(k, 3, 2)$. The result follows by induction from the case $G_S = G_{n-r}$. □

We finish by giving a rough estimate of the number of 3-regular graphs for which Conjecture 1 can now be proved.

Lemma 11 *Let $R(n)$ be the number of 3-regular graphs up to isomorphism on $2n$ vertices and let $R'(n)$ be the number of these for which Conjecture 1 holds. Then*

$$\log R(n) = n \log n + O(n)$$

$$\log R'(n) \geq \frac{2n}{3} \log n + O(n)$$

Proof. It is known [7] that

$$R(n) = \frac{(6n)!}{2^{3n} 6^{2n} (3n)!(2n)!}(c + o(1))$$

for some constant $c > 0$. Using Stirling's formula gives

$$\log R(n) = 6n \log 6n - 3n \log 3n - 2n \log 2n + O(n) = n \log n + O(n).$$

Now let us construct a large class of labeled graphs for which Theorem 9 can inductively color G. Start with a (Hamiltonian) cycle $x_1, \dots, x_{2n}$ and partition these vertices into sets V_i, $4 \leq i \leq k$, with each V_i consisting of $2s_i$ consecutive vertices. Assume that $4 \leq 2s_i \leq f(i-1, 2, 0)$ for each $i > 4$ and $s_4 \leq 3$. Fill in chords in each V_i. The number of ways of doing this is at least $(2s_i - 1)(2s_i - 3) \dots 1 = \frac{(2s_i)!}{s_i! 2^{s_i}}$ for each i. The total number of graphs we can obtain is therefore at least $N = \prod \frac{(2s_i)!}{s_i! 2^{s_i}}$. Estimating using Stirling's formula gives

$$\log N = \sum (s_i \log s_i + O(s_i)) = \frac{2n}{3} \log n + O(n).$$

The maximum number of these that can be isomorphic to a given one is at most $2n.3.2^{2n-2}$ since this is the maximum number of directed Hamiltonian paths in these graphs. Hence

$$\log R'(n) \geq \log N - O(n) = \frac{2n}{3} \log n + O(n)$$

$\square$

Although Theorem 9 does not apply to most 3-regular graphs, the idea of removing edges and coloring the resulting graph with $k-1$ colors was used extensively in Result 1 where Conjecture 2 was checked for all 3-regular graphs of degree at most 22.

References

[1] M. Aigner, E. Triesch, and Z. Tuzu, Irregular Assignments and Vertex-Distinguishing Edge-Colorings of Graphs, *Combinatorics* **90**, (A. Barlotti et al, eds.). Elsevier Science Pub., New York (1992) 1–9.

[2] P.N. Balister, Packing circuits into K_n, *Combin. Probab. Comput.* **10** (2001), 463–499.

[3] P.N. Balister, B. Bollobás and R.H. Schelp, Vertex-Distinguishing Colorings of Graphs with $\Delta(G) = 2$, *Disc. Math.* **252** (1–3) (2002) 17–29.

[4] P.N. Balister, A. Kostochka, Hao Li and R.H. Schelp, Balanced edge colorings, submitted.

[5] C. Bazgan, A. Harkat-Benhamdine, Hao Li and M. Woźniak, On the Vertex-Distinguishing Proper Edge-Colorings of Graphs, *J. Combin. Theory B* **75** (1999) 288–301.

[6] B. Bollobás, Modern Graph Theory, *Graduate Texts in Mathematics*, Springer p152–154.

[7] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European J. Combin.* **1** (1980), no. 4, 311–316.

[8] A.C. Burris and R.H. Schelp, Vertex-Distinguishing Proper Edge-Colourings, *J. Graph Theory* **26** (2) (1997) 70–82.

[9] J. Černý, M. Hořnák, and R. Soták, Observability of a Graph, *Math. Slovaca* **46** (1996), 21–31.

[10] M. Hořnák and R. Soták, Observability of Complete Multipartite Graphs with Equipotent Parts, *Ars Combin.* **41** (1995) 289–301.

[11] M. Hořnák and R. Soták, Asymptotic Behaviour of the Observability of Q_n, *Discrete Math.* **176** (1997) 139–148.