Title: Techniques in Combinatorics (Math 7038/8038)

Catalog Description:
Isoperimetric inequalities: the MYBL inequality, the Kruskal-Katona theorem, edge and vertex isoperimetric inequalities; Talagrand's inequality; The EKR theorem and exact intersection theorems; Martingale inequalities and the chromatic number of a random graph; Entropy inequalities and their applications; Correlation inequalities, including Harris, FKJ, van den Berg-Kesten inequalities; Influence of random variables and sharp threshold results: the KKL theorem and its consequences. Prerequisites: permission of instructor.

Instructor: Béla Bollobás

Textbooks: Material will be provided by the instructor.

Course outline:

The aim of the course is to introduce the student to a wide variety of techniques used in combinatorics: counting, real analytic, combinatorial, Fourier analytic, probabilistic.

Thus, all the basic isoperimetric inequalities, including the MYBL inequality and the Kruskal-Katona inequality, will be proved by counting, compressions, and other clever rearrangements of the sets in question.

The more sophisticated isoperimetric inequality of Talagrand will be proved by techniques of real analysis, by first defining an unusual metric. This inequality has some applications that seem to be out of reach by other means.

Algebraic techniques will be illustrated by proving some theorems of Fisher, Frankl and Wilson, and Grolmusz concerning the question of how large a family of sets can be if the size of the intersections modulo an integer m is specified. In addition, a number of surprising applications will be given.

Basic probabilistic techniques, based on the use of expectation and second moment, will be used to prove a number of important results about random graphs, and to prove the existence of unexpected structures, like a graph of large girth and large chromatic number.

Some martingale inequalities will be proved that enable one to go way beyond what we can prove by moments only. One of the applications concerns the chromatic number of random graphs.

Basic correlation inequalities, including Harris's lemma, the FKG inequality and the van den Berg-Kesten lemma, will be proved by a mixture of combinatorial and analytic techniques. Numerous applications will be given.

Basic probabilistic techniques are used to prove LLL, the Lovász Local Lemma. The great importance of this lemma will be demonstrated by a number of applications. Connections with polynomials will lead to Shearer's theorem, telling us the limitations of LLL, and the classical theorem of Dobrushin in statistical mechanics.

The simplest properties of entropy will be used to deduce the fundamental theorem of Balister-Bollobás, which has numerous instant consequences, including Shearer's theorem on entropy, and a number of inequalities concerning sum-sets.

Fourier analytic techniques combined with an operator inequality will enable us to prove the theorem of Kahn, Kalai, Linial, Bourgain and Katznelson about the influence of random variables. From there, it is but a short step to the theorem of Fridgut and Kalai about sharp thresholds.

Prerequisites: mathematical maturity, familiarity with basic ideas of probability theory and Fourier analysis, and preferably at least one graduate level course on combinatorics. The lecturer may give permission to dispense with the latter.

Course Requirements: Grades (A-F) will depend on attendance and participation in class. Students enrolled at the 8000 level will be expected to present selected papers in class.